

Generalized Jacquet modules of parabolic induction

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ABSTRACT. In this paper we study a generalization of the Jacquet module of a parabolic induction and construct a filtration on it. The successive quotient of the filtration is written by using the twisting functor.

§1. Introduction

The Jacquet module of a representation of a semisimple (or reductive) Lie group is introduced by Casselman [Cas80]. One of the motivation of considering the Jacquet module is to investigate homomorphisms to principal series representations, which is an important invariant of a representation.

One of the powerful tools to study the Jacquet module of a parabolic induction is the Bruhat filtration [CHM00]. This is a filtration on the Jacquet module defined from the Bruhat decomposition. Casselman-Hecht-Milicic [CHM00] use the Bruhat filtration to determine the dimension of the (moderate-growth) Whittaker model of a principal series representation (another proof for Kostant's result). In this paper, we study the Bruhat filtration and show that the successive quotient is described using the twisting functor defined by Arkhipov [Ark04]. If a principal series representation has the unique Langlands quotient, then the successive quotient is the induction from the Jacquet module of a smaller group (a Levi part of a parabolic subgroup). However, in a general case, it becomes “twisted” induction, which has the same character as that of an induced representation but has a different module structure.

Moreover, we investigate its generalization, this is related to the Whittaker model. In [Cas80], Casselman suggested to generalize the notion of the Jacquet module. For this generalized Jacquet module, we can also define

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a Bruhat filtration and the successive quotient of the resulting filtration is described in terms of the generalized twisting functor.

This result gives a strategy to determine all Whittaker models of a parabolic induction. To determine it, it is sufficient to study the successive quotients and extensions of the filtration. In a special case, we can carry out these steps.

Now we state our results precisely. Let G be a connected semisimple Lie group, $G = KA_0N_0$ an Iwasawa decomposition and $P_0 = M_0A_0N_0$ a minimal parabolic subgroup and its Langlands decomposition. As usual, the complexification of the Lie algebra is denoted by the corresponding German letter (for example, $\mathfrak{g} = \text{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$). Fix a character η of N_0 . Then for a representation V of G , the generalized Jacquet modules $J'_\eta(V)$ and $J^*_\eta(V)$ are defined as follows.

Definition 1.1. *Let V be a finite-length moderate growth Fréchet representation of G (See Casselman [Cas89, pp. 391]). We define \mathfrak{g} -modules $J'_\eta(V)$ and $J^*_\eta(V)$ by*

$$J'_\eta(V) = \left\{ v \in V' \mid \begin{array}{l} \text{For some } k \text{ and for all } X \in \mathfrak{n}_0, \\ (X - \eta(X))^k v = 0 \end{array} \right\},$$

$$J^*_\eta(V) = \left\{ v \in (V_{K\text{-finite}})^* \mid \begin{array}{l} \text{For some } k \text{ and for all } X \in \mathfrak{n}_0, \\ (X - \eta(X))^k v = 0 \end{array} \right\},$$

where V' is the continuous dual of V .

Let W be the little Weyl group of G and take $w \in W$. Then the generalized twisting functor $T_{w,\eta}$ is defined as follows. Let $\overline{\mathfrak{n}}_0$ be the nil-radical of the opposite parabolic subalgebra to \mathfrak{p}_0 and e_1, \dots, e_l be a basis of $\text{Ad}(w)\overline{\mathfrak{n}}_0 \cap \mathfrak{n}_0$ such that each e_i is a root vector with respect to \mathfrak{h} where \mathfrak{h} is a Cartan subalgebra of \mathfrak{g} which contains \mathfrak{a}_0 . Moreover, we choose e_i such that $\bigoplus_{i \leq j-1} \mathbb{C}e_i$ is an ideal of $\bigoplus_{i \leq j} \mathbb{C}e_i$ for all j . Let $U(\mathfrak{g})$ be the universal enveloping algebra of \mathfrak{g} and $\overline{U}(\mathfrak{g})_{e_i - \eta(e_i)}$ the localization of $U(\mathfrak{g})$ by a multiplicative set $\{(e_i - \eta(e_i))^n \mid n \in \mathbb{Z}_{>0}\}$. Put $S_{w,\eta} = (U(\mathfrak{g})_{e_i - \eta(e_i)} / U(\mathfrak{g})) \otimes_{U(\mathfrak{g})} \cdots \otimes_{U(\mathfrak{g})} (U(\mathfrak{g})_{e_l - \eta(e_l)} / U(\mathfrak{g}))$. Then $S_{w,\eta}$ is a \mathfrak{g} -bimodule. The twisting functor $T_{w,\eta}$ is defined by $T_{w,\eta}V = S_{w,\eta} \otimes_{U(\mathfrak{g})} (wV)$ where wV is a representation twisted by w (i.e., $Xv = \text{Ad}(w)^{-1}(X) \cdot v$ for $X \in \mathfrak{g}$ and $v \in wV$ where dot means the original action).

Let P be a parabolic subgroup containing A_0N_0 and take a Langlands decomposition $P = MAN$ such that $A_0 \supset A$. Define $\rho_0 \in \mathfrak{a}_0^*$ by $\rho_0(H) = (1/2) \text{Tr ad}(H)|_{\mathfrak{n}_0}$. Let ρ be a restriction of ρ_0 on \mathfrak{a} . An element of \mathfrak{a}^* corresponds to a character of A . We denote the corresponding character to

$\lambda + \rho$ by $e^{\lambda+\rho}$ for $\lambda \in \mathfrak{a}^*$. Then for an irreducible representation σ of M and $\lambda \in \mathfrak{a}^*$, the parabolic induction $\text{Ind}_P^G(\sigma \otimes e^{\lambda+\rho})$ is defined. Let W_M be the little Weyl group of M . Define a subset $W(M)$ of W by $W(M) = \{w \in W \mid \text{for all positive restricted root } \alpha \text{ of } M, w(\alpha) \text{ is positive}\}$. Then $W(M)$ is a complete representatives of W/W_M and parameterizes N_0 -orbits in G/P . For $w \in W$, fix a lift in G and denote it by the same letter w . Enumerate $W(M) = \{w_1, \dots, w_r\}$ so that $\bigcup_{j \leq i} N_0 w_j P/P$ is a closed subset of G/P . Using the C^∞ -realization of a parabolic induction, we can regard an element of $J'_\eta(\text{Ind}_P^G(\sigma \otimes e^{\lambda+\rho}))$ as a distribution on G/P . Then the Bruhat filtration $I_i \subset J'_\eta(\text{Ind}_P^G(\sigma \otimes e^{\lambda+\rho}))$ is defined by

$$I_i = \left\{ x \in J'_\eta(\text{Ind}_P^G(\sigma \otimes e^{\lambda+\rho})) \mid \text{supp } x \subset \bigcup_{j \leq i} N_0 w_j P \right\}.$$

Since $w_i \in W(M)$, we have $\text{Ad}(w_i)(\mathfrak{m} \cap \mathfrak{n}_0) \subset \mathfrak{n}_0$. Hence we can define a character $w_i^{-1}\eta$ of $\mathfrak{m} \cap \mathfrak{n}_0$ by $(w_i^{-1}\eta)(X) = \eta(\text{Ad}(w_i)X)$. Using this character, we can define an $\mathfrak{m} \oplus \mathfrak{a}$ -module $J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$. Then we have the following theorem.

Theorem 1.2 (Theorem 4.7, Theorem 6.1). *The filtration $\{I_i\}$ has the following properties.*

- (1) *If the character η is not unitary, then $J'_\eta(\text{Ind}_P^G(\sigma \otimes e^{\lambda+\rho})) = 0$.*
- (2) *Assume that η is unitary. The module I_i/I_{i-1} is nonzero if and only if η is trivial on $w_i N w_i^{-1} \cap N_0$ and $J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho}) \neq 0$.*
- (3) *If $I_i/I_{i-1} \neq 0$ then $I_i/I_{i-1} \simeq T_{w_i, \eta}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho}))$ where \mathfrak{n} acts $J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$ trivially.*

Under the assumptions that P is a minimal parabolic subgroup, σ is the trivial representation, $\text{Ind}_P^G(\sigma \otimes e^{\lambda+\rho})$ has the unique Langlands quotient and η is the trivial representation, this theorem is proved in [Abe]. The proof we give in [Abe] is algebraic, while we give an analytic and geometric proof in this paper.

For a module $J_\eta^*(\text{Ind}_P^G(\sigma \otimes e^{\lambda+\rho}))$, we have the following theorem. We define two functors. For a $U(\mathfrak{g})$ -module V , put $C(V) = ((V^*)_{\mathfrak{h}\text{-finite}})^*$ and $\Gamma_\eta(V) = \{v \in V \mid \text{for some } k \text{ and for all } X \in \mathfrak{n}_0, (X - \eta(X))^k v = 0\}$.

Theorem 1.3 (Theorem 7.3). *There exists a filtration $0 = \tilde{I}_0 \subset \tilde{I}_1 \subset \dots \subset \tilde{I}_r = J_\eta^*(\text{Ind}_P^G(\sigma \otimes \lambda))$ such that $\tilde{I}_i/\tilde{I}_{i-1} \simeq \Gamma_\eta(C(T_{w_i}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} J^*(\sigma \otimes e^{\lambda+\rho})))$ where \mathfrak{n} acts $J^*(\sigma \otimes e^{\lambda+\rho})$ trivially.*

We state an application. The space of Whittaker vectors $\text{Wh}_\eta(D)$ is defined by $\text{Wh}_\eta(D) = \{x \in D \mid (X - \eta(X))x = 0 \text{ for all } X \in \mathfrak{n}_0\}$ for a $U(\mathfrak{g})$ -module D . If V is a moderate-growth Fréchet representation of G , an element of $\text{Wh}_\eta(V')$ corresponds to a moderate-growth homomorphism $V \rightarrow \text{Ind}_{N_0}^G \eta$ and an element of $\text{Wh}_\eta((V_{K\text{-finite}})^*)$ corresponds to an algebraic homomorphism $V_{K\text{-finite}} \rightarrow \text{Ind}_{N_0}^G \eta$. In particular, when η is the trivial representation, these correspond to homomorphisms to principal series representations.

Let Σ (resp. Σ_M) be the restricted root system for (G, A_0) (resp. $(M, M \cap A_0)$), Σ^+ a positive system of Σ corresponding to N_0 and $\Pi \subset \Sigma$ the set of simple roots determined by Σ^+ . Put $\Sigma_M^+ = \Sigma_M \cap \Sigma^+$. Let \widetilde{W} (resp. \widetilde{W}_M) be the (complex) Weyl group of \mathfrak{g} (resp. \mathfrak{m}). Let $\tilde{\mu} \in \mathfrak{h}^*$ be the infinitesimal character of σ . Let Δ be the root system of $(\mathfrak{g}, \mathfrak{h})$. Put $\Sigma_\eta^+ = (\sum_{\eta|_{\mathfrak{g}\beta} \neq 0, \beta \in \Pi} \mathbb{Z}\beta) \cap \Sigma^+$. Fix a W -invariant bilinear form $\langle \cdot, \cdot \rangle$ of \mathfrak{a}_0 . Using the direct decompositions $(\mathfrak{m} \cap \mathfrak{a}_0)^* \oplus \mathfrak{a}^* = \mathfrak{a}_0^*$ and $\mathfrak{a}_0^* \oplus (\mathfrak{h} \cap \mathfrak{m}_0)^* = \mathfrak{h}^*$, we regard $\mathfrak{a}^* \subset \mathfrak{a}_0^* \subset \mathfrak{h}^*$. Recall that $\nu \in (\mathfrak{m} \cap \mathfrak{a}_0)^*$ is called an exponent of σ if $\nu + \rho_0|_{\mathfrak{m} \cap \mathfrak{a}_0}$ is an $(\mathfrak{m} \cap \mathfrak{a}_0)$ -weight of $\sigma/(\mathfrak{m} \cap \mathfrak{n}_0)\sigma$. We prove the following theorem.

Theorem 1.4 (Theorem 8.5, Theorem 8.12). *For $\lambda \in \mathfrak{a}^*$ and an irreducible representation σ of M , the following formulae hold.*

(1) *Assume that for all $w \in W$ such that $\eta|_{wNw^{-1} \cap N_0} = 1$, the following two conditions hold:*

- (a) *For each exponent ν of σ and $\alpha \in \Sigma^+ \setminus w^{-1}(\Sigma_M^+ \cup \Sigma_\eta^+)$, we have $2\langle \alpha, \lambda + \nu \rangle / |\alpha|^2 \notin \mathbb{Z}_{\leq 0}$.*
- (b) *For all $\tilde{w} \in \widetilde{W}$, we have $\lambda - \tilde{w}(\lambda + \tilde{\mu})|_{\mathfrak{a}} \notin \mathbb{Z}_{\leq 0}((\Sigma^+ \setminus \Sigma_M^+) \cap w^{-1}\Sigma^+)|_{\mathfrak{a}} \setminus \{0\}$.*

Then we have

$$\begin{aligned} \dim \text{Wh}_\eta((\text{Ind}_P^G(\sigma \otimes e^{\lambda+\rho}))') \\ = \sum_{w \in W(M), \eta|_{wNw^{-1} \cap N_0} = 1} \dim \text{Wh}_{w^{-1}\eta}(\sigma'). \end{aligned}$$

(2) *Assume that for all $\tilde{w} \in \widetilde{W} \setminus \widetilde{W}_M$ we have $(\lambda + \tilde{\mu}) - \tilde{w}(\lambda + \tilde{\mu}) \notin \mathbb{Z}\Delta$.*

Then we have

$$\begin{aligned} \dim \mathrm{Wh}_\eta((\mathrm{Ind}_P^G(\sigma \otimes e^{\lambda+\rho})_{K\text{-finite}})^*) \\ = \sum_{w \in W(M)} \dim \mathrm{Wh}_{w^{-1}\eta}((\sigma_M \cap K\text{-finite})^*). \end{aligned}$$

In the case that σ is finite-dimensional, we have the following theorem, which have been announced by T. Oshima (cf. his talk at National University of Singapore, January 11, 2006). Let Δ_M be the root system for $(\mathfrak{m} \oplus \mathfrak{a}, \mathfrak{h})$ and take a positive system Δ_M^+ compatible with Σ_M^+ . Put $\widetilde{\rho}_M = (1/2) \sum_{\alpha \in \Delta_M^+} \alpha$. For subsets Θ_1, Θ_2 of Π , put $\Sigma_{\Theta_i} = \mathbb{Z}\Theta_i \cap \Sigma$, $W(\Theta_i) = \{w \in W \mid w(\Theta_i) \subset \Sigma^+\}$, W_{Θ_i} the Weyl group of Σ_{Θ_i} and $W(\Theta_1, \Theta_2) = \{w \in W(\Theta_1) \cap W(\Theta_2)^{-1} \mid w(\Sigma_{\Theta_1}) \cap \Sigma_{\Theta_2} = \emptyset\}$. The parabolic subgroup P defines a subset of Π . We denote this set by Θ .

Theorem 1.5. *Assume that σ is an irreducible finite-dimensional representation with highest weight $\widetilde{\nu}$. Let $\dim_M(\lambda + \widetilde{\nu})$ be the dimension of a finite-dimensional irreducible representation of $M_0 A_0$ with highest weight $\lambda + \widetilde{\nu}$.*

(1) *Let $\widetilde{\nu}$ be the highest weight of σ . Assume that for all $w \in W$ such that $\eta|_{wN_0w^{-1} \cap N_0} = 1$ the following two conditions hold:*

- (a) *For all $\alpha \in \Sigma^+ \setminus w^{-1}(\Sigma_M^+ \cup \Sigma_\eta^+)$ we have $2\langle \alpha, \lambda + w_0 \widetilde{\nu} \rangle / |\alpha|^2 \notin \mathbb{Z}_{\leq 0}$.*
- (b) *For all $\widetilde{w} \in \widetilde{W}$ we have $\lambda - \widetilde{w}(\lambda + \widetilde{\nu} + \widetilde{\rho}_M)|_{\mathfrak{a}} \notin \mathbb{Z}_{\leq 0}((\Sigma^+ \setminus \Sigma_M^+) \cap w^{-1}\Sigma^+)|_{\mathfrak{a}} \setminus \{0\}$.*

Then we have

$$\dim \mathrm{Wh}_\eta(I(\sigma, \lambda)') = \#W(\mathrm{supp} \eta, \Theta) \times (\dim_M(\lambda + \widetilde{\nu}))$$

(2) *Assume that for all $\widetilde{w} \in \widetilde{W} \setminus \widetilde{W}_M$, $(\lambda + \widetilde{\nu}) - \widetilde{w}(\lambda + \widetilde{\nu}) \notin \Delta$. Then we have*

$$\begin{aligned} \dim \mathrm{Wh}_\eta((I(\sigma, \lambda)_{K\text{-finite}})^*) \\ = \#W(\mathrm{supp} \eta, \Theta) \times \#W_{\mathrm{supp} \eta} \times (\dim_M(\lambda + \widetilde{\nu})) \end{aligned}$$

We summarize the content of this paper. In §2, we introduce the Bruhat filtration. From §2 to §6 we study the module $J'_\eta(\mathrm{Ind}_P^G(\sigma \otimes \lambda))$. In §3 we prove the successive quotient is zero under some conditions. The structure

of the successive quotients is investigated in §4. We give the definition and properties of the generalized twisting functor in §5 and, in §6 we reveal the relation between the twisting functor and the successive quotient. We complete the proof of Theorem 1.2 in this section. Theorem 1.3 is proved in §7. In §8, the dimension of the space of Whittaker vectors is determined and Theorem 1.4 and Theorem 1.5 are proved.

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List of Symbols

$\text{supp}_G \eta = \text{supp } \eta$	§2, 8	$I(\sigma, \lambda)$	§2, 9
\mathcal{L}	§2, 9	$W(M)$	§2, 9
r	§2, 9	I_i	§2, 9
U_i	§2, 9	O_i	§2, 9
Res_i	§2, 9	δ_i	§2, 9
$\mathcal{P}(O_i)$	§2, 10	η_i	§2, 10
$D_i(X)$	§2, 11	$R'_i(X)$	§3, 15
$\widehat{R}_i(X)$	§3, 15	$\delta_i(T, f, u')$	§3, 15
L_X	§3, 17	$\text{Wh}_\eta(V)$	§3, 20
$P_\eta = M_\eta A_\eta N_\eta$	§4, 21	$\mathfrak{p}_\eta = \mathfrak{m}_\eta \oplus \mathfrak{a}_\eta \oplus \mathfrak{n}_\eta$	§4, 21
\mathfrak{l}_η	§4, 21	$\overline{N_\eta}$	§4, 21
$\overline{\mathfrak{n}_\eta}$	§4, 21	$\Sigma_\eta^+, \Sigma_\eta^-$	§4, 21
$P_{M,\eta} = M_{M,\eta} A_{M,\eta} N_{M,\eta}$	§4, 21	$\mathfrak{p}_{\mathfrak{m},\eta} = \mathfrak{m}_{\mathfrak{m},\eta} \oplus \mathfrak{a}_{\mathfrak{m},\eta} \oplus \mathfrak{n}_{\mathfrak{m},\eta}$	§4, 21
$\Phi_{w,w'}$	§4, 21	κ	§4, 23
H	§4, 23	$D(X, \lambda)$	§4, 26
$\mathfrak{g}_\alpha^{\mathfrak{h}}$	§5, 27	\mathfrak{u}_0	§5, 27
$\overline{\mathfrak{u}_0}$	§5, 27	$\mathfrak{u}_{0,\tilde{w}}$	§5, 27
$S_{\tilde{w},\psi}$	§5, 27	$S_{e_k - \eta(e_k)}$	§5, 27
$T_{\tilde{w},\psi}$	§5, 28	J_i	§6, 29
$J(V)$	§7, 32	\mathcal{O}'_{P_0}	§7, 32
$\mathcal{O}'_{\overline{P_0}}$	§7, 32	$D'(V)$	§7, 32
$C(V)$	§7, 32	\mathcal{O}'	§7, 32
$\Gamma_\eta(V)$	§7, 32	\tilde{I}_i	§7, 33
$\gamma_1, \gamma_2, \gamma_3, \gamma_4$	§8, 34	$W(\Theta)$	§8, 45
$W(\Theta_1, \Theta_2)$	§8, 45	W_Θ	§8, 45

$\mathcal{D}'(U, \mathcal{L})$ $\S A, 47 \quad \mathcal{T}(M, \mathcal{L})$ $\S A, 47$ **Notation**

Throughout this paper we use the following notation. As usual we denote the ring of integers, the set of non-negative integers, the set of positive integers, the real number field and the complex number field by $\mathbb{Z}, \mathbb{Z}_{\geq 0}, \mathbb{Z}_{> 0}, \mathbb{R}$ and \mathbb{C} , respectively. Let G be a connected semisimple Lie group and \mathfrak{g} the complexification of its Lie algebra. Fix a Cartan involution θ of G and denote its derivation by the same letter θ . Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be the decomposition of \mathfrak{g} into the $+1$ and -1 eigenspaces for θ . Set $K = \{g \in G \mid \theta(g) = g\}$. Let $P_0 = M_0 A_0 N_0$ be a minimal parabolic subgroup and its Langlands decomposition such that $M_0 \subset K$ and $\text{Lie}(A_0) \subset \mathfrak{s}$. Denote the complexification of the Lie algebra of P_0, M_0, A_0, N_0 by $\mathfrak{p}_0, \mathfrak{m}_0, \mathfrak{a}_0, \mathfrak{n}_0$, respectively. Take a parabolic subgroup P which contains P_0 and denote its Langlands decomposition by $P = MAN$. Here we assume $A \subset A_0$. Let $\mathfrak{p}, \mathfrak{m}, \mathfrak{a}, \mathfrak{n}$ be the complexification of the Lie algebra of P, M, A, N . Put $\overline{P}_0 = \theta(P_0)$, $\overline{N}_0 = \theta(N_0)$, $\overline{P} = \theta(P)$, $\overline{N} = \theta(N)$, $\overline{\mathfrak{p}}_0 = \theta(\mathfrak{p}_0)$, $\overline{\mathfrak{n}}_0 = \theta(\mathfrak{n}_0)$, $\overline{\mathfrak{p}} = \theta(\mathfrak{p})$ and $\overline{\mathfrak{n}} = \theta(\mathfrak{n})$.

In general, we denote the dual space $\text{Hom}_{\mathbb{C}}(V, \mathbb{C})$ of a \mathbb{C} -vector space V by V^* . Let $\Sigma \subset \mathfrak{a}_0^*$ be the restricted root system for $(\mathfrak{g}, \mathfrak{a}_0)$ and \mathfrak{g}_{α} the root space for $\alpha \in \Sigma$. Then $\sum_{\alpha \in \Sigma} \mathbb{R}\alpha$ is a real form of \mathfrak{a}_0^* . We denote the real part of $\lambda \in \mathfrak{a}_0^*$ with respect to this real form by $\text{Re } \lambda$ and the imaginary part by $\text{Im } \lambda$. Let Σ^+ be the positive system determined by \mathfrak{n}_0 . Put $\rho_0 = \sum_{\alpha \in \Sigma^+} (\dim \mathfrak{g}_{\alpha}/2)\alpha$ and $\rho = \rho_0|_{\mathfrak{a}}$. The positive system Σ^+ determines the set of simple roots Π . Fix a totally order of $\sum_{\alpha \in \Sigma} \mathbb{R}\alpha$ such that the following conditions hold: (1) If $\alpha > \beta$ and $\gamma \in \sum_{\alpha \in \Sigma} \mathbb{R}\alpha$ then $\alpha + \gamma > \beta + \gamma$. (2) If $\alpha > 0$ and c is a positive real number then $c\alpha > 0$. (3) For all $\alpha \in \Sigma^+$ we have $\alpha > 0$. Write W for the little Weyl group for $(\mathfrak{g}, \mathfrak{a}_0)$, e for the unit element of W and w_0 for the longest element of W . For $w \in W$, we fix a representative in $N_K(\mathfrak{a})$ and denote it by the same letter w .

Let \mathfrak{t}_0 be a Cartan subalgebra of \mathfrak{m}_0 and T_0 the corresponding Cartan subgroup of M_0 . Then $\mathfrak{h} = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ is a Cartan subalgebra of \mathfrak{g} . Let Δ be the root system for $(\mathfrak{g}, \mathfrak{h})$ and take a positive system Δ^+ compatible with Σ^+ , i.e., if $\alpha \in \Delta^+$ satisfies that $\alpha|_{\mathfrak{a}_0} \neq 0$ then $\alpha|_{\mathfrak{a}_0} \in \Sigma^+$. Let $\mathfrak{g}_{\alpha}^{\mathfrak{h}}$ be the root space of $\alpha \in \Delta$ and \widetilde{W} the Weyl group of Δ . Put $\tilde{\rho} = (1/2) \sum_{\alpha \in \Delta^+} \alpha$. By the decompositions $(\mathfrak{m} \cap \mathfrak{a}_0)^* \oplus \mathfrak{a}^* = \mathfrak{a}_0^*$ and $\mathfrak{t}_0^* \oplus \mathfrak{a}_0^* = \mathfrak{h}^*$, we always regard $\mathfrak{a}^* \subset \mathfrak{a}_0^* \subset \mathfrak{h}^*$.

We use the same notation for M , i.e., Σ_M be the restricted root system of M , $\Sigma_M^+ = \Sigma_M \cap \Sigma^+$, W_M the little Weyl group of M , Δ_M the root system of M , $\Delta_M^+ = \Delta_M \cap \Delta^+$, \widetilde{W}_M the Weyl group of M and $w_{M,0}$ the longest element of W_M .

We can define an anti-isomorphism of $U(\mathfrak{g})$ by $X \mapsto -X$ for $X \in \mathfrak{g}$. We write this anti-isomorphism by $u \mapsto \check{u}$.

For a \mathfrak{g} -module V and $g \in G$, we define a \mathfrak{g} -module gV as follows: The representation space is V and the action of $X \in \mathfrak{g}$ is $X \cdot v = (\text{Ad}(g)^{-1}X)v$ for $v \in gV$.

For $\xi = (\xi_1, \dots, \xi_l) \in \mathbb{Z}^l$, put $|\xi| = \xi_1 + \dots + \xi_l$.

§2. Parabolic induction and the Bruhat filtration

Fix a character η of \mathfrak{n}_0 and put $\text{supp}_G \eta = \text{supp } \eta = \{\alpha \in \Pi \mid \eta|_{\mathfrak{g}_\alpha} \neq 0\}$. The character η is called *non-degenerate* if $\text{supp } \eta = \Pi$. We denote the character of N_0 whose differential is η by the same letter η .

Definition 2.1. *Let V be a finite-length moderate growth Fréchet representation of G (See Casselman [Cas89, pp. 391]). We define \mathfrak{g} -modules $J'_\eta(V)$ and $J^*_\eta(V)$ by*

$$J'_\eta(V) = \left\{ v \in V' \mid \begin{array}{l} \text{For some } k \text{ and for all } X \in \mathfrak{n}_0, \\ (X - \eta(X))^k v = 0 \end{array} \right\},$$

$$J^*_\eta(V) = \left\{ v \in (V_{K\text{-finite}})^* \mid \begin{array}{l} \text{For some } k \text{ and for all } X \in \mathfrak{n}_0, \\ (X - \eta(X))^k v = 0 \end{array} \right\},$$

where V' is the continuous dual of V .

Put $J'(V) = J'_0(V)$ and $J^*(V) = J^*_0(V)$ where 0 is the trivial representation of \mathfrak{n}_0 . The module $J^*(V)$ is the (dual of) *Jacquet module* defined by Casselman [Cas80]. By the automatic continuation theorem [Wal83, Theorem 4.8], we have $J'(V) = J^*(V)$. The correspondence $V \mapsto J'_\eta(V)$ and $V \mapsto J^*_\eta(V)$ are functors from the category of G -modules to the category of \mathfrak{g} -modules.

In this section, we study the module $J'_\eta(V)$ for a parabolic induction V . An element of \mathfrak{a}^* is identified with a character of A . We denote the character of A corresponding to $\lambda + \rho$ by $e^{\lambda+\rho}$ where $\lambda \in \mathfrak{a}^*$. For an irreducible moderate growth Fréchet representation σ of M and $\lambda \in \mathfrak{a}^*$, put

$$I(\sigma, \lambda) = C^\infty\text{-Ind}_P^G(\sigma \otimes e^{\lambda+\rho}).$$

(For a moderate growth Fréchet representation, see Casselman [Cas89].) The representation $I(\sigma, \lambda)$ has a natural structure of a moderate growth Fréchet representation. Denote its continuous dual by $I(\sigma, \lambda)'$.

Let \mathcal{L} be a vector bundle on G/P attached to the representation $\sigma \otimes e^{\lambda+\rho}$ and \mathcal{L}' be the continuous dual vector bundle of \mathcal{L} .

REMARK 2.2. A C^∞ -section of \mathcal{L} corresponds to a σ -valued C^∞ -function f on G such that $f(gman) = \sigma(m)^{-1} e^{-(\lambda+\rho)(\log a)} f(g)$ for $g \in G$, $m \in M$, $a \in A$, $n \in N$. In particular a C^∞ -function on G/P corresponds to a right P -invariant C^∞ -function on G . We use this identification throughout this paper.

We use the notation in Appendix A. We can regard $J'_\eta(I(\sigma, \lambda))$ as a subspace of $\mathcal{D}'(G/P, \mathcal{L})$. Set $W(M) = \{w \in W \mid w(\Sigma_M^+) \subset \Sigma^+\}$. Then it is known that the multiplication map $W(M) \times W_M \rightarrow W$ is bijective [Kos61, Proposition 5.13]. By the Bruhat decomposition, we have

$$G/P = \bigsqcup_{w \in W(M)} N_0 w P/P.$$

(Recall that we fix a representative of $w \in W$, see Notation.) Enumerate $W(M) = \{w_1, \dots, w_r\}$ so that $\bigcup_{j \leq i} N_0 w_j P/P$ is a closed subset of G/P for each i . Then we can define a submodule I_i of $J'_\eta(I(\sigma, \lambda))$ by

$$I_i = \left\{ x \in J'_\eta(I(\sigma, \lambda)) \mid \text{supp } x \subset \bigcup_{j \leq i} N_0 w_j P/P \right\}.$$

The filtration $\{I_i\}$ is called the Bruhat filtration [CHM00]. In the rest of this section, we study the module I_i/I_{i-1} . Put $U_i = w_i \overline{N} P/P$ and $O_i = N_0 w_i P/P$. The subset U_i is an open subset of G/P containing O_i and $U_i \cap O_j = \emptyset$ if $j < i$. Hence, the restriction map $\text{Res}_i: I_i \rightarrow \mathcal{D}(U_i, \mathcal{L})$ induces an injective map $\text{Res}_i: I_i/I_{i-1} \rightarrow \mathcal{D}(U_i, \mathcal{L})$. Moreover, $\text{Im } \text{Res}_i \subset \mathcal{T}_{O_i}(U_i, \mathcal{L})$. We have $\mathcal{T}_{O_i}(U_i, \mathcal{L}) = U(\text{Ad}(w_i) \overline{n} \cap \overline{n}) \otimes_{\mathbb{C}} \mathcal{T}(O_i, \mathcal{L}|_{O_i})$ by Proposition A.2.

Notice that by a map $n \mapsto n w_i P/P$ we have isomorphisms $w_i \overline{N} w_i^{-1} \simeq U_i$ and $w_i \overline{N} w_i^{-1} \cap N_0 \simeq O_i$. Fix a Haar measure on $w_i \overline{N} w_i^{-1} \cap N_0$. Then we can define $\delta_i \in \mathcal{D}'(O_i, \mathcal{L}|_{O_i})$ by

$$\langle \delta_i, \varphi \rangle = \int_{w_i \overline{N} w_i^{-1} \cap N_0} \varphi(n w_i) dn.$$

for $\varphi \in C_c^\infty(O_i, \mathcal{L}|_{O_i})$. By the exponential map $\text{Ad}(w_i) \text{Lie}(\overline{N}) \rightarrow w_i \overline{N} w_i^{-1}$ and diffeomorphism $w_i \overline{N} w_i^{-1} \simeq U_i$, U_i has a vector space structure and

O_i is a subspace of U_i . Let $\mathcal{P}(O_i)$ be the ring of polynomials on O_i (cf. Appendix A.3 or [CG90]). Define a C^∞ -function η_i on O_i by $\eta_i(nw_iP/P) = \eta(n)$ for $n \in w_i\bar{N}w_i^{-1} \cap N_0$. For a C^∞ -function f on O_i and $u' \in \sigma'$, we define $f \otimes u' \in C^\infty(O_i, \sigma')$ by $(f \otimes u')(x) = f(x)u'$. Since $w_i \in W(M)$, $\text{Ad}(w_i)(\mathfrak{m} \cap \mathfrak{n}_0) \subset \mathfrak{n}_0$. Hence we can define a character $w_i^{-1}\eta$ of $\mathfrak{m} \cap \mathfrak{n}_0$ by $(w_i^{-1}\eta)(X) = \eta(\text{Ad}(w_i)X)$. Using this character, we can define the Jacquet module $J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$ of MA -representation $\sigma \otimes e^{\lambda+\rho}$. This is an $\mathfrak{m} \oplus \mathfrak{a}$ -module. Put

$$I'_i = \left\{ \sum_{k=1}^l T_k(((f_k \eta_i^{-1}) \otimes u'_k) \delta_i) \mid \begin{array}{l} T_k \in U(\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \bar{\mathfrak{n}}), \quad f_k \in \mathcal{P}(O_i), \\ u'_k \in J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho}) \end{array} \right\}.$$

The space I'_i is a $U(\mathfrak{g})$ -submodule of $\mathcal{D}'(U_i, \mathcal{L})$. Our aim is to prove that if i satisfies some conditions then $I_i/I_{i-1} \simeq I'_i$.

Lemma 2.3. *Let E_1, \dots, E_n be a basis of $\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \bar{\mathfrak{n}}_0$ such that each E_s is a restricted root vector for some root (say α_s) and $F \in (\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0) \oplus \text{Ad}(w_i)(\mathfrak{m} \cap \mathfrak{n}_0)$. (Notice that $\text{Ad}(w_i)(\mathfrak{m} \cap \mathfrak{n}_0) = \text{Ad}(w_i)\mathfrak{m} \cap \mathfrak{n}_0$ since $w_i \in W(M)$, so $(\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0) \oplus \text{Ad}(w_i)(\mathfrak{m} \cap \mathfrak{n}_0) = \text{Ad}(w_i)(\bar{\mathfrak{n}} \oplus \mathfrak{m}) \cap \mathfrak{n}_0$ is a subalgebra of \mathfrak{g} .) For $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{Z}_{\geq 0}^n$, set $E^\xi = E_1^{\xi_1} E_2^{\xi_2} \dots E_n^{\xi_n}$. Then for all $c \in \mathbb{C}$ we have*

$$\begin{aligned} [(F - c)^k, E^\xi] &\in \left(\sum_{\xi' \in A(\xi)} \mathbb{C} E^{\xi'} \right) U((\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0) \oplus \text{Ad}(w_i)(\mathfrak{m} \cap \mathfrak{n}_0)) \\ &\subset U(\text{Ad}(w_i)(\bar{\mathfrak{n}} \oplus (\mathfrak{m} \cap \mathfrak{n}_0))) \end{aligned}$$

where $A(\xi) = \{\xi' \in \mathbb{Z}_{\geq 0}^n \mid |\xi'| < |\xi|, \text{ or } |\xi'| = |\xi| \text{ and } \sum \xi'_i \alpha_i < \sum \xi_i \alpha_i\}$.

PROOF. We may assume $k = 1$. We prove the lemma by induction on $|\xi|$. We have

$$[F - c, E^\xi] = [F, E^\xi] = \sum_{s=1}^n \sum_{l=0}^{\xi_s-1} E_1^{\xi_1} \dots E_{s-1}^{\xi_{s-1}} E_s^l [F, E_s] E_s^{\xi_s-l-1} E_{s+1}^{\xi_{s+1}} \dots E_n^{\xi_n}.$$

Hence, it is sufficient to prove

$$\begin{aligned} E_1^{\xi_1} \dots E_{s-1}^{\xi_{s-1}} E_s^l [F, E_s] E_s^{\xi_s-l-1} E_{s+1}^{\xi_{s+1}} \dots E_n^{\xi_n} \\ \in \left(\sum_{\xi' \in A(\xi)} \mathbb{C} E^{\xi'} \right) U((\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0) \oplus \text{Ad}(w_i)(\mathfrak{m} \cap \mathfrak{n}_0)). \end{aligned}$$

We may assume that F is a restricted root vector. If $[F, E_s] \in \text{Ad}(w_i)\bar{\mathfrak{n}} \cap \bar{\mathfrak{n}}_0$ then the claim hold. Assume that $[F, E_s] \in (\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0) \oplus \text{Ad}(w_i)(\mathfrak{m} \cap \mathfrak{n}_0)$. Put $\xi^{(1)} = (\xi_1, \dots, \xi_{s-1}, l, 0, \dots, 0) \in \mathbb{Z}^n$ and $\xi^{(2)} = (0, \dots, 0, \xi_s - l - 1, \xi_{s+1}, \dots, \xi_n) \in \mathbb{Z}^n$. Using inductive hypothesis, we have

$$\begin{aligned} & E^{\xi^{(1)}} [[F, E_s], E^{\xi^{(2)}}] \\ & \in E^{\xi^{(1)}} \left(\sum_{\xi' \in A(\xi^{(2)})} \mathbb{C} E^{\xi'} \right) U((\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0) \oplus \text{Ad}(w_i)(\mathfrak{m} \cap \mathfrak{n}_0)) \\ & \subset \left(\sum_{\xi' \in A(\xi^{(1)} + \xi^{(2)})} \mathbb{C} E^{\xi'} \right) U((\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0) \oplus \text{Ad}(w_i)(\mathfrak{m} \cap \mathfrak{n}_0)) \\ & \subset \left(\sum_{\xi' \in A(\xi)} \mathbb{C} E^{\xi'} \right) U((\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0) \oplus \text{Ad}(w_i)(\mathfrak{m} \cap \mathfrak{n}_0)) \end{aligned}$$

On the other hand, we have

$$\begin{aligned} E^{\xi^{(1)}} E^{\xi^{(2)}} [F, E_s] & \in \left(\sum_{|\xi'| \leq |\xi^{(1)} + \xi^{(2)}|} \mathbb{C} E^{\xi'} \right) [F, E_s] \\ & \subset \left(\sum_{|\xi'| \leq |\xi^{(1)} + \xi^{(2)}|} \mathbb{C} E^{\xi'} \right) (\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0) \oplus \text{Ad}(w_i)(\mathfrak{m} \cap \mathfrak{n}_0). \end{aligned}$$

Since $|\xi^{(1)} + \xi^{(2)}| = |\xi| - 1 < |\xi|$, we get the lemma. \square

Let X be an element of the normalizer of $\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0$ in \mathfrak{g} . For $f \in C^\infty(O_i)$ we define $D_i(X)f \in C^\infty(O_i)$ by

$$(D_i(X)f)(nw_i) = \left. \frac{d}{dt} f(\exp(-tX)n \exp(tX)w_i) \right|_{t=0}$$

where $n \in w_i \bar{N} w_i^{-1} \cap N_0$.

Lemma 2.4. Fix $f \in C^\infty(O_i)$, $u' \in (\sigma \otimes e^{\lambda+\rho})'$ and $X \in \mathfrak{g}$.

(1) If $X \in \mathfrak{a}_0$, then X normalizes $\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0$ and we have

$$\begin{aligned} X((f \otimes u')\delta_i) & = ((D_i(X)f) \otimes u')\delta_i + (f \otimes ((\text{Ad}(w_i)^{-1}X)u'))\delta_i \\ & \quad + (w_i \rho_0 - \rho_0)(X)(f \otimes u')\delta_i. \end{aligned}$$

(2) If $X \in \text{Ad}(w_i)(\mathfrak{m} \cap \mathfrak{n}_0)$ or $X \in \mathfrak{m}_0$, then X normalizes $\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0$ and we have

$$X((f \otimes u')\delta_i) = ((D_i(X)f) \otimes u')\delta_i + (((\text{Ad}(w_i)^{-1}X)u') \otimes f)\delta_i.$$

PROOF. Let X be as in the lemma. First we prove that X normalizes $\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0$. If $X \in \mathfrak{m}_0 + \mathfrak{a}_0$, then X normalizes each restricted root space. Hence, X normalizes $\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0$. If $X \in \text{Ad}(w_i)(\mathfrak{m} \cap \mathfrak{n}_0)$, then $X \in \mathfrak{n}_0$ since $w_i \in W(M)$. Hence, X normalizes \mathfrak{n}_0 . Since \mathfrak{m} normalizes $\bar{\mathfrak{n}}$, X normalizes $\text{Ad}(w_i)\bar{\mathfrak{n}}$.

Put $g_t = \exp(tX)$. Take $\varphi \in C_c^\infty(U_i, \mathcal{L})$ and we regard φ as a σ -valued C^∞ -function on $w_i\bar{N}P$ (Remark 2.2). Since $w_i g_t w_i^{-1} \in P$, we have $\varphi(xw_i g_t w_i^{-1}) = \sigma(w_i g_t w_i^{-1})^{-1}\varphi(x)$. Put $D(t) = |\det(\text{Ad}(g_t)^{-1}|_{\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0})|$. Then

$$\begin{aligned} \langle X((f \otimes u')\delta_i), \varphi \rangle &= \langle (f \otimes u')\delta_i, -X\varphi \rangle \\ &= \frac{d}{dt} \int_{w_i\bar{N}w_i^{-1} \cap N_0} u'(\varphi(g_t n w_i)) f(n w_i) dn \Big|_{t=0} \\ &= \frac{d}{dt} \int_{w_i\bar{N}w_i^{-1} \cap N_0} u'(\varphi((g_t n g_t^{-1}) w_i (w_i^{-1} g_t w_i))) f(n w_i) dn \Big|_{t=0} \\ &= \frac{d}{dt} \int_{w_i\bar{N}w_i^{-1} \cap N_0} u'(\sigma(w_i^{-1} g_t w_i)^{-1} \varphi((g_t n g_t^{-1}) w_i)) f(n w_i) dn \Big|_{t=0} \\ &= \frac{d}{dt} \int_{w_i\bar{N}w_i^{-1} \cap N_0} u'(\sigma(w_i^{-1} g_t w_i)^{-1} \varphi(n w_i)) f(g_t^{-1} n g_t w_i) D(t) dn \Big|_{t=0} \\ &= \frac{d}{dt} \int_{w_i\bar{N}w_i^{-1} \cap N_0} ((w_i^{-1} g_t w_i) u')(\varphi(n w_i)) f(g_t^{-1} n g_t w_i) D(t) dn \Big|_{t=0} \end{aligned}$$

This implies

$$\begin{aligned} X((f \otimes u')\delta_i) &= ((D_i(X)f) \otimes u')\delta_i + (f \otimes ((\text{Ad}(w_i)^{-1}X)u'))\delta_i \\ &\quad + \frac{d}{dt} |\det(\text{Ad}(g_t)^{-1}|_{\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}})| \Big|_{t=0} ((f \otimes u')\delta_i) \end{aligned}$$

(1) Assume that $X \in \mathfrak{a}_0$. Since $w_i \in W(M)$, we have $w_i\bar{N}w_i^{-1} \cap N_0 = w_i\bar{N}_0w_i^{-1} \cap N_0$. This implies that $\det(\text{Ad}(g_t)^{-1}|_{\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0}) = e^{t(w_i\rho_0 - \rho_0)(X)}$.

(2) First assume that $X \in \mathfrak{m}_0$. Since $g \mapsto \det(\text{Ad}(g)^{-1}|_{\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0})$ is 1-dimensional representation, it is unitary since M_0 is compact. Hence we

have $|\det(\text{Ad}(g_t)^{-1}|_{\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0})| = 1$. Next assume that $X \in (\text{Ad}(w_i)\mathfrak{m} \cap \mathfrak{n}_0)$. Then $\text{ad}(X)$ is nilpotent. Hence, $\text{Ad}(g_t) - 1$ is nilpotent. This implies $\det(\text{Ad}(g_t)^{-1}|_{\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0}) = 1$. \square

Lemma 2.5. *Let $x \in \mathcal{T}_{O_i}(U_i, \mathcal{L})$. Assume that for all $X \in \text{Ad}(w_i)\bar{\mathfrak{p}} \cap \mathfrak{n}_0$ there exists a positive integer k such that $(X - \eta(X))^k x = 0$. Then $x \in I'_i$. In particular we have $\text{Im Res}_i \subset I'_i$.*

PROOF. Let E_s and α_s be as in Lemma 2.3. For $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{Z}_{\geq 0}^n$, set $E^\xi = E_1^{\xi_1} E_2^{\xi_2} \dots E_n^{\xi_n}$. Since $x \in \mathcal{T}_{O_i}(U_i, \mathcal{L}) = U(\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0) \otimes \mathcal{T}(O_i, \mathcal{L})$, there exist $x_\xi \in \mathcal{T}(O_i, \mathcal{L})$ such that $x = \sum_\xi E^\xi x_\xi$ (finite sum).

First we prove $x_\xi \in (\mathcal{P}(O_i)\eta_i^{-1} \otimes (\sigma \otimes e^{\lambda+\rho})')\delta_i$ by backward induction on the lexicological order of $(|\xi|, \sum_s \xi_s \alpha_s)$. Fix a nonzero element $F \in \text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0$. Then $(F - \eta(F))^k x = \sum_\xi [(F - \eta(F))^k, E^\xi](x_\xi) + \sum_\xi E^\xi ((F - \eta(F))^k x_\xi)$. Assume that $(F - \eta(F))^k x = 0$. Define the set $A(\xi)$ as in Lemma 2.3. By Lemma 2.3, we have

$$\begin{aligned} \sum_\xi E^\xi ((F - \eta(F))^k x_\xi) &= - \sum_\xi [(F - \eta(F))^k, E^\xi](x_\xi) \\ &\in \sum_\xi \left(\sum_{\xi' \in A(\xi)} \mathbb{C} E^{\xi'} \right) U((\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}) \oplus \text{Ad}(w_i)(\mathfrak{m} \cap \mathfrak{n}_0))(x_\xi). \end{aligned}$$

Put $B(\xi) = \{\xi' \mid |\xi'| > |\xi| \text{ or } |\xi'| = |\xi| \text{ and } \sum \xi'_s \alpha_s > \sum \xi_s \alpha_s\}$. Notice that $U((\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}) \oplus \text{Ad}(w_i)(\mathfrak{m} \cap \mathfrak{n}_0))(x_\xi) \in \mathcal{T}(O_i, \mathcal{L})$. Since $\mathcal{T}_{O_i}(U_i, \mathcal{L}) = U(\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \bar{\mathfrak{n}}) \otimes_{\mathbb{C}} \mathcal{T}(O_i, \mathcal{L}|_{O_i})$, we have

$$(F - \eta(F))^k x_\xi \in \sum_{\xi' \in B(\xi)} U((\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}) \oplus \text{Ad}(w_i)(\mathfrak{m} \cap \mathfrak{n}_0))(x_{\xi'}).$$

By inductive hypothesis, $x_{\xi'} \in (\mathcal{P}(O_i)\eta_i^{-1} \otimes (\sigma \otimes e^{\lambda+\rho})')\delta_i$ for all $\xi' \in B(\xi)$. Hence we have $(F - \eta(F))^k x_\xi \in (\mathcal{P}(O_i)\eta_i^{-1} \otimes (\sigma \otimes e^{\lambda+\rho})')\delta_i$. Therefore $x_\xi \in (\mathcal{P}(O_i)\eta_i^{-1} \otimes (\sigma \otimes e^{\lambda+\rho})')\delta_i$ by Corollary A.4.

Hence, we can write $x = \sum_\xi E^\xi \sum_l (f_{\xi,l} \eta_i^{-1} \otimes u'_{\xi,l}) \delta_i$ (finite sum), where $f_{\xi,l} \in \mathcal{P}(O_i)$ and $u'_{\xi,l} \in (\sigma \otimes e^{\lambda+\rho})'$. Moreover, we can assume that $f_{\xi,l}$ is an \mathfrak{a}_0 -weight vector with respect to D_i and $\{f_{\xi,l}\}_l$ is lineally independent for each ξ . We prove $u'_{\xi,l} \in J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$. Take $F \in \mathfrak{n}_0 \cap \mathfrak{m}$. By Lemma 2.4,

we have

$$\begin{aligned}
& (\text{Ad}(w_i)F - \eta(\text{Ad}(w_i)F))^k x \\
&= \sum_{\xi, l} [(\text{Ad}(w_i)F - \eta(\text{Ad}(w_i)F))^k, E^\xi] ((f_{\xi, l} \eta_i^{-1} \otimes u'_{xi, l}) \delta_i) \\
&\quad + \sum_{\xi} E^\xi \sum_{p=0}^k \binom{k}{p} (((D_i(\text{Ad}(w_i)F))^{k-p} (f_{\xi, l} \eta_i^{-1}) \otimes \\
&\quad (F - \eta(\text{Ad}(w_i)F))^p (u'_{\xi, l})) \delta_i.
\end{aligned}$$

Now we prove $u'_{\xi, l} \in J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$ by backward induction on the lexicological order of $(|\xi|, \sum \xi_s \alpha_s, -\text{wt } f_{\xi, l})$ where $\text{wt } f_{\xi, l}$ is an \mathfrak{a}_0 -weight of $f_{\xi, l}$ with respect to D_i . Take k such that $(\text{Ad}(w_i)F - \eta(\text{Ad}(w_i)F))^k x = 0$. Then we have

$$\begin{aligned}
& f_{\xi, l} \otimes (F - \eta(\text{Ad}(w_i)F))^k (u'_{\xi, l}) \delta_i \\
&\in \sum_{\eta \in B(\xi), l} U((\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0) \oplus \text{Ad}(w_i)(\mathfrak{m} \cap \mathfrak{n}_0)) ((f_{\eta, l} \eta_i^{-1} \otimes u'_{\eta, l}) \delta_i) \\
&\quad + \sum_{\text{wt } f_{\eta, l'} < \text{wt } f_{\xi, l}} \sum_p (((D_i(\text{Ad}(w_i)F))^p f_{\eta, l'} \eta_i^{-1}) \otimes (U(\mathbb{C}F) u'_{\eta, l'})) \delta_i.
\end{aligned}$$

By inductive hypothesis, we have $(F - \eta(F))^k u'_{\xi, l} \in J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$. This implies that $u'_{\xi, l} \in J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$. \square

In fact, we have $\text{Im Res}_i = I'_i$ under some conditions. This is proved in Section 4.

§3. Vanishing theorem

In this section, we fix $i \in \{1, 2, \dots, r\}$ and a basis $\{e_1, e_2, \dots, e_l\}$ of $\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0$. Here we assume that each e_i is a restricted root vector and denote its root by α_i .

By the decomposition

$$\begin{aligned}
N_0/[N_0, N_0] &\simeq ((w_i \bar{P} w_i^{-1} \cap N_0)/(w_i \bar{P} w_i^{-1} \cap [N_0, N_0])) \\
&\quad \times ((w_i N w_i^{-1} \cap N_0)/(w_i N w_i^{-1} \cap [N_0, N_0]))
\end{aligned}$$

where $[\cdot, \cdot]$ is the commutator group, we can define a character η' of N_0 by $\eta'(n) = \eta(n)$ for $n \in w_i \bar{P} w_i^{-1} \cap N_0$ and $\eta'(n) = 1$ for $n \in w_i N w_i^{-1} \cap N_0$.

Lemma 3.1. *Let $X \in \mathfrak{n}_0$. Then for all $x \in I'_i$ there exists a positive integer k such that $(X - \eta'(X))^k x = 0$.*

To prove this lemma, we prepare some notation. For $X \in \text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0$, we define a differential operator $R'_i(X)$ on O_i by

$$(R'_i(X)\varphi)(nw_iP/P) = \left. \frac{d}{dt} \varphi(n \exp(tX)w_iP/P) \right|_{t=0}$$

where $n \in w_i\bar{N}w_i^{-1} \cap N_0$. (Recall that $w_i\bar{N}w_i^{-1} \cap N_0 \simeq O_i$ by the map $n \mapsto nw_iP/P$.)

For $X \in \mathfrak{g}$, we define a differential operator $\tilde{R}_i(X)$ on $w_i\bar{N}P$ by the same way, i.e., for a C^∞ -function φ on $w_i\bar{N}P$, put

$$(\tilde{R}_i(X)\varphi)(pw_i) = \left. \frac{d}{dt} \varphi(p \exp(tX)w_i) \right|_{t=0}$$

for $p \in w_i\bar{N}Pw_i^{-1}$. Notice that even if φ is right P -invariant, $\tilde{R}_i(X)\varphi$ is not right P -invariant in general.

Since R'_i (resp. \tilde{R}_i) is a Lie algebra homomorphism, we can define a differential operator $R'_i(T)$ (resp. $\tilde{R}_i(T)$) for $T \in U(\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0)$ (resp. $T \in U(\mathfrak{g})$) as usual. For $T \in U(\mathfrak{g})$, $f \in C^\infty(O_i)$ and $u' \in (\sigma \otimes (\lambda + \rho))'$, we define $\delta_i(T, f, u') \in \mathcal{D}'_{O_i}(U_i, \mathcal{L})$ by

$$\langle \delta_i(T, f, u'), \varphi \rangle = \int_{w_i\bar{N}w_i^{-1} \cap N_0} f(nw_i) u'((\tilde{R}_i(T)\varphi)(nw_i)) dn$$

where $\varphi \in C_c^\infty(U_i, \mathcal{L})$ and we regard φ as a function on $w_i\bar{N}P$ (Remark 2.2). The following lemma is easy to prove.

Lemma 3.2. *We have the following properties.*

- (1) For $X \in \text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0$, $\delta_i(XT, f, u') = \delta_i(T, R'_i(-X)(f), u')$.
- (2) For $X \in \text{Ad}(w_i)\mathfrak{p}$, $\delta_i(TX, f, u') = \delta_i(T, f, \text{Ad}(w_i)^{-1}Xu')$.
- (3) The map $C^\infty(O_i) \otimes_{U(\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n})} U(\mathfrak{g}) \otimes_{U(\text{Ad}(w_i)\mathfrak{p})} w_i(\sigma \otimes e^{\lambda+\rho})' \rightarrow \mathcal{D}'(U_i, O_i, \mathcal{L})$ defined by $f \otimes T \otimes u' \mapsto \delta_i(T, f, u')$ is injective.

Lemma 3.3. *Let $\{e_i\}$ be a basis of $\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0$ such that e_i is a restricted root vector, α_i the restricted root for e_i , $T, T' \in U(\mathfrak{g})$, $f \in C^\infty(O_i)$*

and $u' \in (\sigma \otimes e^{\lambda+\rho})'$. Then we have

$$\begin{aligned} T\delta_i(T', f, u') \\ = \sum_{(k_1, \dots, k_l) \in \mathbb{Z}_{\geq 0}^l} \delta_i \left((\text{ad}(e_l)^{k_l} \cdots \text{ad}(e_1)^{k_1} T) T', f \prod_{s=1}^l \frac{(-x_s)^{k_s}}{k_s!}, u' \right), \end{aligned}$$

where x_i is a polynomial on O_i given by $\exp(a_1 e_1) \cdots \exp(a_l e_l) w_i P / P \mapsto a_i$. (Notice that the right hand side is a finite sum since $\text{ad}(e_i)$ is nilpotent.)

PROOF. We remark that by a map $(a_1, \dots, a_l) \mapsto \exp(a_1 e_1) \cdots \exp(a_l e_l)$, we have a diffeomorphism $\mathbb{R}^l \simeq w_i \bar{N} w_i^{-1} \cap N_0$ and a Haar measure of $w_i \bar{N} w_i^{-1} \cap N_0$ corresponds to the Euclidean measure of \mathbb{R}^l . Take $\varphi \in C_c^\infty(w_i \bar{N} P, \sigma \otimes e^{\lambda+\rho})$. Put $n(a) = \exp(a_1 e_1) \cdots \exp(a_l e_l)$ for $a = (a_1, \dots, a_l)$. Recall the definition of \tilde{T} from Notation. For $T \in \mathfrak{g}$, we have

$$\begin{aligned} \langle T\delta_i(T', f, u'), \varphi \rangle \\ = \int_{\mathbb{R}^l} u'((\tilde{T} \tilde{R}_i(T') \varphi)(n(a) w_i)) f(n(a) w_i) da \\ = \frac{d}{dt} \int_{\mathbb{R}^l} u'(\tilde{R}_i(T') \varphi)(\exp(tT) n(a) w_i)) f(n(a) w_i) da \Big|_{t=0} \\ = \frac{d}{dt} \int_{\mathbb{R}^l} u'((\tilde{R}_i(T') \varphi)(n(a) \exp(t \text{Ad}(n(a))^{-1} T) w_i)) f(n(a) w_i) da \Big|_{t=0}. \end{aligned}$$

The formula

$$\begin{aligned} \text{Ad}(n(a))^{-1} T &= e^{-\text{ad}(a_l e_l)} \cdots e^{-\text{ad}(a_1 e_1)} T \\ &= \sum_{(k_1, \dots, k_l) \in \mathbb{Z}_{\geq 0}^l} \frac{(-a_1)^{k_1}}{k_1!} \cdots \frac{(-a_l)^{k_l}}{k_l!} \text{ad}(e_l)^{k_l} \cdots \text{ad}(e_1)^{k_1} T \end{aligned}$$

gives the lemma. \square

For $\mathbf{k} = (k_1, \dots, k_l)$, we denote an operator $\text{ad}(e_l)^{k_l} \cdots \text{ad}(e_1)^{k_1}$ on \mathfrak{g} by $\text{ad}(e)^{\mathbf{k}}$ and a function $((-x_1)^{k_1}/k_1!) \cdots ((-x_l)^{k_l}/k_l!) \in \mathcal{P}(O_i)$ by $f_{\mathbf{k}}$.

Lemma 3.4. *Let $\mathbf{k} = (k_1, \dots, k_l) \in \mathbb{Z}_{\geq 0}^l$ and $X \in \mathfrak{n}_0$. Assume that $\text{ad}(e)^{\mathbf{k}} X \in \text{Ad}(w_i) \bar{\mathfrak{n}} \cap \mathfrak{n}_0$. Then we have $R'_i(\text{ad}(e)^{\mathbf{k}} X) f_{\mathbf{k}} = 0$.*

PROOF. We may assume that X is a restricted root vector and denote its restricted root by α . We consider an \mathfrak{a}_0 -weight with respect to D_i . An \mathfrak{a}_0 -weight of $f_{\mathbf{k}}$ is $-\sum_s k_s \alpha_s$. This implies that $R'_i(\text{ad}(e)^{\mathbf{k}} X) f_{\mathbf{k}}$ has an \mathfrak{a}_0 -weight

α . However, $\mathcal{P}(O_i)$ has a decomposition into the direct sum of \mathfrak{a}_0 -weight spaces and its weight belongs to $\{\sum_{\beta \in \Sigma^+} b_\beta \beta \mid b_\beta \in \mathbb{Z}_{\leq 0}\}$. Hence, we have $R'_i(\text{ad}(e)^{\mathbf{k}}X)f_{\mathbf{k}} = 0$. \square

For $f \in \mathcal{P}(O_i)$ and $X \in \mathfrak{n}_0$ we define $L_X(f)$ by

$$L_X(f)(nw_i) = \left. \frac{d}{dt} f(\exp(-tX)nw_i) \right|_{t=0}.$$

Lemma 3.5. *Let $X \in \mathfrak{n}_0$ be a restricted root vector. For $f \in \mathcal{P}(O_i)$ and $u' \in J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$, we have*

$$\begin{aligned} (X - \eta'(X))\delta_i(1, f\eta_i^{-1}, u') &= \delta_i(1, L_X(f)\eta_i^{-1}, u') \\ + \sum_{\text{ad}(e)^{\mathbf{k}}X \in \text{Ad}(w_i)\mathfrak{n}_0 \cap \mathfrak{n}_0} \delta_i(1, f f_{\mathbf{k}}\eta_i^{-1}, (\text{Ad}(w_i)^{-1}(\text{ad}(e)^{\mathbf{k}}X) - \eta'(\text{ad}(e)^{\mathbf{k}}X))u'). \end{aligned}$$

(Again the sum of the right hand side is finite.)

PROOF. We have

$$X\delta_i(1, f\eta_i^{-1}, u') = \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^l} \delta_i(\text{ad}(e)^{\mathbf{k}}X, f f_{\mathbf{k}}\eta_i^{-1}, u').$$

by Lemma 3.3. Since $\text{ad}(e)^{\mathbf{k}}X$ belongs to \mathfrak{n}_0 and is a restricted root vector, we have either $\text{ad}(e)^{\mathbf{k}}X \in \text{Ad}(w_i)\overline{\mathfrak{n}_0} \cap \mathfrak{n}_0$ or $\text{ad}(e)^{\mathbf{k}}X \in \text{Ad}(w_i)\mathfrak{n}_0 \cap \mathfrak{n}_0$. Recall that $\text{Ad}(w_i)\overline{\mathfrak{n}_0} \cap \mathfrak{n}_0 = \text{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0$ since $w_i \in W(M)$. Assume that $\text{ad}(e)^{\mathbf{k}}X \in \text{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0$. By the definition of η_i and η' , we have $R'_i(-\text{ad}(e)^{\mathbf{k}}X)(\eta_i^{-1}) = \eta(\text{ad}(e)^{\mathbf{k}}X)\eta_i^{-1} = \eta'(\text{ad}(e)^{\mathbf{k}}X)\eta_i^{-1}$. Hence, using Lemma 3.4,

$$\begin{aligned} &\delta_i(\text{ad}(e)^{\mathbf{k}}X, f f_{\mathbf{k}}\eta_i^{-1}, u') \\ &= \delta_i(1, R'_i(-\text{ad}(e)^{\mathbf{k}}X)(f f_{\mathbf{k}}\eta_i^{-1}), u') \\ &= \delta_i(1, R'_i(-\text{ad}(e)^{\mathbf{k}}X)(f) f_{\mathbf{k}}\eta_i^{-1}, u') + \eta'(\text{ad}(e)^{\mathbf{k}}X)\delta_i(1, f f_{\mathbf{k}}\eta_i^{-1}, u'). \end{aligned}$$

Next assume that $\text{ad}(e)^{\mathbf{k}}X \in \text{Ad}(w_i)\mathfrak{n}_0 \cap \mathfrak{n}_0$. For $h \in \mathcal{P}(O_i)$, define $\tilde{h} \in \mathcal{P}(U_i)$ by $\tilde{h}(nn_0w_iP) = h(nw_iP)$ for $n \in w_i\overline{N}w_i^{-1} \cap N_0$ and $n_0 \in w_i\overline{N}w_i^{-1} \cap \overline{N_0}$. Then we have $\widetilde{R'_i(Y)h} = \widetilde{\tilde{R}_i(Y)\tilde{h}}$ for all $Y \in \text{Ad}(w_i)\overline{\mathfrak{n}_0} \cap \mathfrak{n}_0$. Since $\tilde{f}(pnw_i) = \tilde{f}(pw_i)$ for $p \in w_i\overline{N}Pw_i^{-1}$ and $n \in w_iN_0w_i^{-1} \cap N_0$, we have

$\widetilde{R}_i(-\mathrm{ad}(e)^{\mathbf{k}}X)(\widetilde{f}) = 0$. Hence we have

$$\begin{aligned} \delta_i(\mathrm{ad}(e)^{\mathbf{k}}X, f f_{\mathbf{k}}\eta_i^{-1}, u') &= \delta_i(1, f f_{\mathbf{k}}\eta_i^{-1}, \mathrm{Ad}(w_i)^{-1}(\mathrm{ad}(e)^{\mathbf{k}}X)u') \\ &= \delta_i(1, R'_i(-\mathrm{ad}(e)^{\mathbf{k}}X)(\widetilde{f})|_{O_i} f_{\mathbf{k}}\eta_i^{-1}, u') + \\ &\quad \delta_i(1, f f_{\mathbf{k}}\eta_i^{-1}, \mathrm{Ad}(w_i)^{-1}(\mathrm{ad}(e)^{\mathbf{k}}X)u'). \end{aligned}$$

By the same calculation as the proof of Lemma 3.3, we have

$$\widetilde{L_X(f)} = L_X(\widetilde{f}) = \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^l} \widetilde{R}_i(-\mathrm{ad}(e)^{\mathbf{k}}X)(\widetilde{f})\widetilde{f}_{\mathbf{k}}.$$

Hence

$$\begin{aligned} \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^l} \delta_i(1, \widetilde{R}_i(-\mathrm{ad}(e)^{\mathbf{k}}X)(\widetilde{f})|_{O_i} f_{\mathbf{k}}\eta_i^{-1}, u') &= \delta_i(1, \widetilde{L_X(f)}|_{O_i} \eta_i^{-1}, u') \\ &= \delta_i(1, L_X(f)\eta_i^{-1}, u'). \end{aligned}$$

These imply that

$$\begin{aligned} (X - \eta'(X))\delta_i(1, f\eta_i^{-1}, u') &= \delta_i(1, L_X(f)\eta_i^{-1}, u') \\ &+ \sum_{\mathrm{ad}(e)^{\mathbf{k}}X \in \mathrm{Ad}(w_i)\mathfrak{n}_0 \cap \mathfrak{n}_0} \delta_i(1, f f_{\mathbf{k}}\eta_i^{-1}, \mathrm{Ad}(w_i)^{-1}(\mathrm{ad}(e)^{\mathbf{k}}X)u') \\ &+ \sum_{\mathrm{ad}(e)^{\mathbf{k}}X \in \mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0} \eta'(\mathrm{ad}(e)^{\mathbf{k}}X)\delta_i(1, f f_{\mathbf{k}}\eta_i^{-1}, u') \\ &\quad - \eta'(X)\delta_i(1, f\eta_i^{-1}, u'). \end{aligned}$$

Since η' is a character, if $\mathbf{k} \neq (0, \dots, 0)$ then $\eta'(\mathrm{ad}(e)^{\mathbf{k}}X) = 0$. Hence we have

$$\sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^n} \eta'(\mathrm{ad}(e)^{\mathbf{k}}X)\delta_i(1, f f_{\mathbf{k}}\eta_i^{-1}, u') = \eta'(X)\delta_i(1, f\eta_i^{-1}, u').$$

This implies

$$\begin{aligned} &\left(\sum_{\mathrm{ad}(e)^{\mathbf{k}}X \in \mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0} \eta'(\mathrm{ad}(e)^{\mathbf{k}}X)\delta_i(1, f f_{\mathbf{k}}\eta_i^{-1}, u') \right) - \eta'(X)\delta_i(1, f\eta_i^{-1}, u') \\ &= \sum_{\mathrm{ad}(e)^{\mathbf{k}}X \in \mathrm{Ad}(w_i)\mathfrak{n}_0 \cap \mathfrak{n}_0} \eta'(\mathrm{ad}(e)^{\mathbf{k}}X)\delta_i(1, f f_{\mathbf{k}}\eta_i^{-1}, u'). \end{aligned}$$

We get the lemma. \square

PROOF OF LEMMA 3.1. Since $\text{ad}(\mathfrak{n}_0)$ acts \mathfrak{g} nilpotently, the subspace

$$\{x \in I'_i \mid \text{for some } k \text{ and for all } X \in \mathfrak{n}_0, (X - \eta(X))^k x = 0\}$$

is \mathfrak{g} -stable. Hence we may assume that $x = ((f\eta_i^{-1}) \otimes u')\delta_i = \delta_i(1, f\eta_i^{-1}, u')$ for some $f \in \mathcal{P}(O_i)$ and $u' \in J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$.

Set $V = U(\text{Ad}(w_i)^{-1}\mathfrak{n}_0 \cap \mathfrak{n}_0)u'$ where \mathfrak{n} acts $J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$ trivially.

Then V is finite-dimensional. By applying Engel's theorem for $V \otimes (-w_i^{-1}\eta')$, there exists a filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_p = V$ such that $(V_s/V_{s-1}) \otimes (-w_i^{-1}\eta'|_{\text{Ad}(w_i)^{-1}\mathfrak{n}_0 \cap \mathfrak{n}_0})$ is the trivial representation of $\text{Ad}(w_i)^{-1}\mathfrak{n}_0 \cap \mathfrak{n}_0$. Then we have $V_s/V_{s-1} \simeq w_i^{-1}\eta'|_{\text{Ad}(w_i)^{-1}\mathfrak{n}_0 \cap \mathfrak{n}_0}$ for all $s = 1, 2, \dots, p$. We prove the lemma by induction on $p = \dim V$.

We may assume that X is a restricted root vector. By Lemma 3.5, we have

$$(X - \eta'(X))\delta_i(1, f\eta_i^{-1}, u') \in \delta_i(1, L_X(f)\eta_i^{-1}, u') + \sum_{h \in \mathcal{P}(O_i), v' \in V_{p-1}} \delta_i(1, h\eta_i^{-1}, v').$$

Since f is a polynomial, there exists a positive integer c such that $(L_X)^c(f) = 0$. Then $(X - \eta'(X))^c \delta_i(1, f\eta_i^{-1}, u') \in \sum_{h \in \mathcal{P}(O_i), v' \in V_{p-1}} \delta_i(1, h\eta_i^{-1}, v')$. By inductive hypothesis the lemma is proved. \square

From the lemma, we get the following vanishing theorem. Recall that we define the character $w_i^{-1}\eta$ of $\mathfrak{m} \cap \mathfrak{n}_0$ by $(w_i^{-1}\eta)(X) = \eta(\text{Ad}(w_i)X)$.

Lemma 3.6. *Assume that $I_i/I_{i-1} \neq 0$. Then the following conditions hold.*

- (1) *The character η is unitary.*
- (2) *The character η is zero on $\text{Ad}(w_i)\mathfrak{n} \cap \mathfrak{n}_0$.*
- (3) *The module $J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$ is not zero.*

PROOF. (2) By Lemma 3.1 and the definition of J'_η , if $I_i/I_{i-1} \neq 0$ then $\eta = \eta'$. By the definition of η' , $\eta = \eta'$ is equivalent to $\eta|_{\text{Ad}(w_i)\mathfrak{n} \cap \mathfrak{n}_0} = 0$.

(3) This is clear from Lemma 2.5.

(1) It is sufficient to prove that if η is not unitary then $J'_\eta(V) = 0$ for all irreducible representation V of G . By Casselman's subrepresentation theorem, V is a subrepresentation of some principal series representation. Since

J'_η is an exact functor, we may assume V is a principal series representation $\text{Ind}_{P_0}^G(\sigma_0 \otimes e^{\lambda_0 + \rho_0})$.

Take the Bruhat filtration $\{I_i\}$ of $J'_\eta(V)$. We prove $I_i/I_{i-1} = 0$ for all i . By (2), if η is non-trivial on $w_i N_0 w_i^{-1} \cap N_0$ then $I_i/I_{i-1} = 0$. Hence we may assume that η is not unitary on $w_i \overline{N_0} w_i^{-1} \cap N_0$. In this case, a nonzero element of I'_i is not tempered. Hence $I_i/I_{i-1} = 0$. \square

REMARK 3.7. In the next section it is proved that the conditions of Lemma 3.6 is also sufficient (Theorem 4.7).

Definition 3.8 (Whittaker vectors). *Let V be a $U(\mathfrak{n}_0)$ -module. We define a vector space $\text{Wh}_\eta(V)$ by*

$$\text{Wh}_\eta(V) = \{v \in V \mid \text{for all } X \in \mathfrak{n}_0 \text{ we have } Xv = \eta(X)v\}.$$

An element of $\text{Wh}_\eta(V)$ is called a Whittaker vector.

Lemma 3.9. *Assume that $\eta|_{\text{Ad}(w_i)\mathfrak{n} \cap \mathfrak{n}_0} = 0$. Then we have*

$$\begin{aligned} \text{Wh}_\eta \left(\left\{ \sum_s (f_s \eta_i^{-1} \otimes u'_s) \delta_i \mid f_s \in \mathcal{P}(O_i), u'_s \in J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho}) \right\} \right) \\ = \{(\eta_i^{-1} \otimes u') \delta_i \mid u'_s \in \text{Wh}_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})\}. \end{aligned}$$

PROOF. By the assumption, we have $\eta = \eta'$. Hence the right hand side is a subspace of the left hand side by Lemma 3.5.

Take $x = \sum_s (f_s \eta_i^{-1} \otimes u'_s) = \sum_s \delta(1, f_s \eta_i^{-1}, u'_s) \in \text{Wh}_\eta(I'_i)$. We assume that $\{u'_s\}$ is linearly independent. Take $X \in \text{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0$. Since $\text{ad}(e)^{\mathbf{k}}X \in \text{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0$ for all $\mathbf{k} \in \mathbb{Z}_{\geq 0}^l$, we have $\sum_s \delta_i(1, L_X(f_s) \eta_i^{-1}, u'_s) = 0$ by Lemma 3.5. Hence $L_X(f_s) = 0$. This implies $f_s \in \mathbb{C}$.

From the above argument, $x = \delta(1, \eta_i^{-1}, u')$ for some $u' \in J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$. Take $X \in \text{Ad}(w_i)\mathfrak{m} \cap \mathfrak{n}_0$. By Lemma 3.5, we have

$$\delta_i(1, \eta_i^{-1}, (\text{Ad}(w_i)^{-1}X - \eta(X))u') \in \sum_{\mathbf{k} \neq 0, u_{\mathbf{k}} \in J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})} \delta_i(1, f_{\mathbf{k}} \eta_i^{-1}, u_{\mathbf{k}}).$$

If $\mathbf{k} \neq 0$ then the degree of $f_{\mathbf{k}}$ is greater than 0. So the left hand side must be 0. Hence we have $(\text{Ad}(w_i)^{-1}X - \eta(X))u' = 0$. We have the lemma. \square

The following lemma is well-known, but we give a proof for the readers (cf. Casselman-Hecht-Milicic [CHM00], Yamashita [Yam86]).

Lemma 3.10. *Assume that $\text{supp } \eta = \Pi$. Let $x \in \text{Wh}_\eta(I(\sigma, \lambda)')$. Then there exists $u' \in \text{Wh}_{w_r^{-1}\eta}((\sigma \otimes e^{\lambda+\rho})')$ such that $x = (\eta_r^{-1} \otimes u')\delta_r$.*

Recall that $r = \#W(M) = \#(W/W_M)$.

PROOF. Assume that $i < r$. Then $w_i w_{M,0}$ is not the longest element of W . There exists a simple root $\alpha \in \Pi$ such that $s_\alpha w_i w_{M,0} > w_i w_{M,0}$. This means that $w_i w_{M,0} \Sigma^+ \cap \Sigma^+ = s_\alpha (s_\alpha w_i w_{M,0} \Sigma^+ \cap \Sigma^+) \cup \{\alpha\}$. The left hand side is $w_i(\Sigma^+ \setminus \Sigma_M^+) \cap \Sigma^+$. Hence, η is not trivial on $\text{Ad}(w_i)\mathfrak{n} \cap \mathfrak{n}_0$. By Lemma 3.6, $I_i/I_{i-1} = 0$. This implies that $J'_\eta(I(\sigma, \lambda)) \subset I'_r$. There exists a polynomial $f_s \in \mathcal{P}(X_r)$ and $u'_s \in J'_{w_r^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$ such that $x = \sum_s ((f_s \eta_r^{-1}) \otimes u'_s) \delta_r$. By Lemma 3.9, we have the lemma. \square

§4. Analytic continuation

The aim of this section is to prove that $\text{Im Res}_i = I'_i$ if $I_i/I_{i-1} \neq 0$.

Let P_η be the parabolic subgroup corresponding to $\text{supp } \eta \subset \Pi$ containing P_0 and $P_\eta = M_\eta A_\eta N_\eta$ its Langlands decomposition such that $A_\eta \subset A_0$. Denote the complexification of the Lie algebra of P_η , M_η , A_η , N_η by \mathfrak{p}_η , \mathfrak{m}_η , \mathfrak{a}_η , \mathfrak{n}_η , respectively. Put $\mathfrak{l}_\eta = \mathfrak{m}_\eta \oplus \mathfrak{a}_\eta$, $\overline{N}_\eta = \theta(N_\eta)$ and $\overline{\mathfrak{n}}_\eta = \theta(\mathfrak{n}_\eta)$. Set $\Sigma_\eta^+ = \{\sum_{\alpha \in \text{supp } \eta} n_\alpha \alpha \in \Sigma^+ \mid n_\alpha \in \mathbb{Z}_{\geq 0}\}$ and $\Sigma_\eta^- = -\Sigma_\eta^+$. The same notation will be used for M with suffix M , i.e., $P_{M,\eta} = M_{M,\eta} A_{M,\eta} N_{M,\eta}$ is the parabolic subgroup of M containing $M \cap P_0$ corresponding to $\text{supp } \eta \cap \Sigma_M^+$, $\mathfrak{p}_{\mathfrak{m},\eta} = \mathfrak{m}_{\mathfrak{m},\eta} \oplus \mathfrak{a}_{\mathfrak{m},\eta} \oplus \mathfrak{n}_{\mathfrak{m},\eta}$ is a complexification of the Lie algebra of $P_{M,\eta} = M_{M,\eta} A_{M,\eta} N_{M,\eta}$.

For $w \in W$, there is an open dense subset $w\overline{N}P/P$ of G/P and it is diffeomorphic to \overline{N} . Then for $w, w' \in W$, there exists a map $\Phi_{w,w'}$ from some open dense subset $U \subset \overline{N}$ to \overline{N} such that $w\overline{n}P/P = w'\Phi_{w,w'}(\overline{n})P/P$ for $\overline{n} \in U$. The map $\Phi_{w,w'}$ is a rational function.

Since the exponential map $\exp: \text{Lie}(\overline{N}) \rightarrow \overline{N}$ is diffeomorphism, \overline{N} has a structure of a vector space.

Lemma 4.1. (1) *The map $\overline{N} \rightarrow \mathbb{R}$ defined by $\overline{n} \mapsto e^{8\rho(H(\overline{n}))}$ is a polynomial.*

(2) *For all $\overline{n} \in \overline{N}$ we have $e^{8\rho(H(\overline{n}))} \geq 1$.*

(3) *Take $H_0 \in \mathfrak{a}$ such that $\alpha(H_0) = -1$ for all $\alpha \in \Pi \setminus \Sigma_M$. There exists a continuous function $Q(\overline{n}) \geq 0$ on \overline{N} such that the following conditions hold: (a) The function Q vanishes only at the unit element. (b) $e^{8\rho(H(\overline{n}))} \geq Q(\overline{n})$. (c) $Q(\exp(tH_0)\overline{n}\exp(-tH_0)) \geq e^{8t}Q(\overline{n})$ for $t \in \mathbb{R}_{>0}$ and $\overline{n} \in \overline{N}$.*

PROOF. By Knapp [Kna01, Proposition 7.19], there exists an irreducible finite-dimensional $V_{4\rho}$ of \mathfrak{g} with the highest weight $4\rho \in \mathfrak{a}_0^* \subset \mathfrak{h}^*$. Let $v_{4\rho} \in V_{4\rho}$ be a highest weight vector and $v_{-4\rho}^* \in V_{4\rho}^*$ the lowest weight vector of $V_{4\rho}^*$. Then $\mathbb{C}v_{4\rho}$ is a 1-dimensional unitary representation of M . Take $\bar{n} \in \bar{N}$ and decompose $\bar{n} = kan$ where $k \in K$, $a \in A_0$ and $n \in N_0$.

First we prove (1). We have $\theta(\bar{n})^{-1}\bar{n} = \theta(n)^{-1}a^2n$. Hence

$$\begin{aligned} \langle \theta(\bar{n})^{-1}\bar{n}v_{4\rho}, v_{-4\rho}^* \rangle &= \langle \theta(n)^{-1}a^2nv_{4\rho}, v_{-4\rho}^* \rangle \\ &= \langle a^2nv_{4\rho}, \theta(n)v_{-4\rho}^* \rangle \\ &= e^{8\rho(H(\bar{n}))} \langle v_{4\rho}, v_{-4\rho}^* \rangle. \end{aligned}$$

The left hand side is a polynomial.

Next we prove (2) and (3). Fix a compact real form of \mathfrak{g} containing $\text{Lie}(K)$ and take an inner product on $V_{4\rho}$ which is invariant under this compact real form. We normalize an inner product $\|\cdot\|$ so that $\|v_{4\rho}\| = 1$. Then we have $\|\bar{n}v_{4\rho}\| = \|kanv_{4\rho}\| = \|av_{4\rho}\| = e^{4\rho(H(\bar{n}))}\|v_{4\rho}\| = e^{4\rho(H(\bar{n}))}$. For $\nu \in \mathfrak{h}^*$ let $Q_\nu(\bar{n}) \in V_{4\rho}$ be the ν -weight vector such that $\bar{n}v_{4\rho} = \sum_\nu Q_\nu(\bar{n})$. Then we have $e^{8\rho(H(\bar{n}))} = \sum_\nu \|Q_\nu(\bar{n})\|^2$. Since $Q_{4\rho}(\bar{n}) = v_{4\rho}$, we have $e^{8\rho(H(\bar{n}))} \geq 1$.

Put $Q(\bar{n}) = \sum_{w \in W(M) \setminus \{e\}} \|Q_{4w\rho}(\bar{n})\|^2$. Assume that $\bar{n} \neq e$. Then there exist $w \in W(M) \setminus \{e\}$, $m' \in M$, $a' \in A$, $n' \in N$ and $\bar{n}' \in \bar{N}$ such that $\bar{n} = w\bar{n}'m'a'n'$. Let $v_{-4w\rho}^* \in V_{4\rho}^*$ be a weight vector with \mathfrak{h} -weight $-4w\rho$. Then we have

$$\begin{aligned} \|Q_{4w\rho}(\bar{n})\| &= |\langle \bar{n}v_{4\rho}, v_{-4w\rho}^* \rangle| = |\langle w\bar{n}'m'a'n'v_{4\rho}, v_{-4w\rho}^* \rangle| \\ &= |\langle a'v_{4\rho}, w^{-1}v_{-4w\rho}^* \rangle| = e^{4\rho(\log a')} |\langle v_{4\rho}, w^{-1}v_{-4w\rho}^* \rangle| \neq 0. \end{aligned}$$

Hence, if $\bar{n} \in \bar{N} \setminus \{e\}$ then $Q(\bar{n}) \neq 0$.

Let t be a positive real number. Using $Q_\nu(\exp(tH_0)\bar{n}\exp(-tH_0)) = e^{t(\nu-4\rho)(H_0)}Q_\nu(\bar{n})$, we have

$$Q(\exp(tH_0)\bar{n}\exp(-tH_0)) = \sum_{w \in W(M) \setminus \{e\}} e^{8t(w\rho-\rho)(H_0)} |Q_{4w\rho}(\bar{n})|^2.$$

Since $(w\rho - \rho)(H_0) \geq 1$ for $w \in W(M) \setminus \{e\}$, we get the lemma. \square

REMARK 4.2. The condition Lemma 4.1 (3) implies that $\lim_{\bar{n} \rightarrow \infty} Q(\bar{n}) = \infty$. The proof is the following. Take H_0 as in Lemma 4.1. Let $\{e_1, \dots, e_l\}$ be a basis of $\bar{\mathfrak{n}}$. Here, we assume that each e_i is a restricted root vector and denote its root by α_i . Any $\bar{n} \in \bar{N}$ can be written as $\bar{n} = \exp(\sum_{i=1}^l a_i e_i)$

where $a_i \in \mathbb{R}$. Put $r(\bar{n}) = \sum_{i=1}^l |a_i|^{-1/\alpha_i(H_0)}$. Set $C = \min_{r(\bar{n})=1} Q(\bar{n})$. Since $Q(\bar{n}) > 0$ if \bar{n} is not the unit element, $C > 0$. Then we have $Q(\bar{n}) \geq Cr(\bar{n})^8$ if $r(\bar{n}) > 1$. If $\bar{n} \rightarrow \infty$ then $r(\bar{n}) \rightarrow \infty$. Hence, $Q(\bar{n}) \rightarrow \infty$.

Lemma 4.3. *Let f be a polynomial on \bar{N} . There exists a positive integer k and a C^∞ -function h on G/P such that $h(w_i \bar{n} P/P) = e^{-k\rho(H(\bar{n}))} f(\bar{n})$ for all $\bar{n} \in \bar{N}$.*

PROOF. By Lemma 4.1 and Remark 4.2, we can choose a positive integer C such that $e^{-8C\rho(H(\bar{n}))} f(\bar{n}) \rightarrow 0$ when $\bar{n} \rightarrow \infty$. Let \tilde{f} be a function on U_i defined by $\tilde{f}(w_i \bar{n} P/P) = e^{-8C\rho(H(\bar{n}))} f(\bar{n})$ for $\bar{n} \in \bar{N}$. We prove that \tilde{f} can be extended to G/P . Take $w \in W(M)$. Then \tilde{f} is defined in a subset of $w\bar{N}P/P$. Using a diffeomorphism $\bar{N} \simeq w\bar{N}P/P$, \tilde{f} defines a rational function $\tilde{f} \circ \Phi_{w_i, w}$ defined on an open dense subset of \bar{N} . By the condition of C , the function $\tilde{f} \circ \Phi_{w_i, w}$ has no pole. Hence, \tilde{f} defines a C^∞ -function on $w\bar{N}P/P$. Since $\bigcup_{w \in W(M)} w\bar{N}P/P = G/P$, the lemma follows. \square

Define $\kappa: G \rightarrow K$ and $H: G \rightarrow \text{Lie}(A_0)$ by $g \in \kappa(g) \exp H(g) N_0$. Recall that for a representation V of \mathfrak{g} , $\nu \in \mathfrak{a}_0^*$ is called an exponent of V if $\nu + \rho_0|_{\mathfrak{m} \cap \mathfrak{a}_0}$ is an \mathfrak{a}_0 -weight of $V/\mathfrak{n}_0 V$.

Proposition 4.4. *Let φ be a σ -valued function on K which satisfies $\varphi(km) = \sigma(m)^{-1} \varphi(k)$ for all $k \in K$ and $m \in M \cap K$. We define $\varphi_\lambda \in I(\sigma, \lambda)$ by $\varphi_\lambda(kman) = e^{-(\lambda+\rho)(\log a)} \sigma(m)^{-1} \varphi(k)$ for $k \in K$, $m \in M$, $a \in A$ and $n \in N$. For $u' \in J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$ and $f \in \mathcal{P}(O_i)$, put $I_{f, u'}(\varphi_\lambda) = \int_{w_i \bar{N} w_i^{-1} \cap N_0} u'(\varphi_\lambda(nw_i)) \eta(n)^{-1} f(nw_i) dn$. (If $\text{supp } \varphi \subset K \cap w_i \bar{N} P$ then the integral converges.)*

- (1) *If $\langle \alpha, \text{Re } \lambda \rangle$ is sufficiently large for each $\alpha \in \Sigma^+ \setminus \Sigma_M^+$ then the integral $I_{f, u'}(\varphi_\lambda)$ absolutely converges.*
- (2) *As a function of λ , the integral $I_{f, u'}(\varphi_\lambda)$ has a meromorphic continuation to \mathfrak{a}^* .*
- (3) *If $\text{supp } \eta = \Pi$ and $i = r$ then $I_{f, u'}(\varphi_\lambda)$ is holomorphic for all $\lambda \in \mathfrak{a}^*$.*
- (4) *Let ν be an exponent of σ and $u' \in \text{Wh}_{w_i^{-1}\eta}((\sigma \otimes e^{\lambda+\rho})')$. If $2\langle \alpha, \lambda + \nu \rangle / |\alpha|^2 \notin \mathbb{Z}_{\leq 0}$ for all $\alpha \in \Sigma^+ \setminus w_i^{-1}(\Sigma^+ \cup \Sigma_\eta^-)$ then $I_{1, u'}(\varphi_\mu)$ is holomorphic at $\mu = \lambda$.*

PROOF. First we prove (1). If $f = 1$ then this is a well-known result. For a general f , extends f to a function on $w_i \bar{N} P/P$ by $f(w_i n n') = f(w_i n)$

for $n \in w_i \overline{N} w_i^{-1} \cap N_0$ and $n' \in w_i \overline{N} w_i^{-1} \cap \overline{N}_0$. Then by Lemma 4.3 there exists a positive number C such that $\overline{n} \mapsto e^{-C\rho(H(\overline{n}))} f(w_i \overline{n})$ extends to a function h on G/P . Since

$$I_{f,u'}(\varphi_\lambda) = \int_{w_i \overline{N} w_i^{-1} \cap N_0} u'(\varphi(\kappa(nw))) e^{-(\lambda+\rho)(H(nw_r))} f(nw_r) \eta(n)^{-1} dn,$$

we have $I_{f,u'}(\varphi_\lambda) = I_{1,u'}((\varphi h)_{\lambda-C\rho})$.

We prove (3). By dualizing Casselman's subrepresentation theorem, there exist a representation σ_0 of M_0 and $\lambda_0 \in \mathfrak{a}_0^*$ such that σ is a quotient of $\text{Ind}_{M \cap P_0}^M(\sigma_0 \otimes e^{\lambda_0})$. Then we may regard $u' \in J'_{w_r^{-1}\eta}(\text{Ind}_{M \cap P_0}^M(\sigma_0 \otimes e^{\lambda_0}))$. By the proof of Lemma 3.10, there exist a polynomial f_0 on $(M \cap N_0)w_{M,0}(M \cap P_0)/(M \cap P_0)$ and $u'_0 \in (\sigma_0 \otimes e^{\lambda_0})'$ such that u' is given by

$$\varphi_0 \mapsto \int_{M \cap N_0} u'_0(\varphi_0(n_0 w_{M,0})) f_0(n_0 w_{M,0}) \eta(n_0)^{-1} dn_0$$

Let $\pi: \text{Ind}_{P_0}^G(\sigma_0 \otimes e^{\lambda+\lambda_0+\rho}) \rightarrow I(\sigma, \lambda)$ be the map induced from the quotient map $\text{Ind}_{M \cap P_0}^M(\sigma_0 \otimes e^{\lambda_0}) \rightarrow \sigma$. Take $\tilde{\varphi}: K \rightarrow \sigma_0$ which satisfies $\tilde{\varphi}(km) = \sigma_0^{-1}(m)\tilde{\varphi}(k)$ ($k \in K, m \in M_0$) and $\pi(\tilde{\varphi}_{\lambda+\lambda_0}) = \varphi_\lambda$. Define a polynomial $\tilde{f} \in \mathcal{P}(w_i w_{M,0} \overline{N}_0 P_0 / P_0)$ by

$$\tilde{f}(w_i w_{M,0} n n_0 P_0 / P_0) = f(w_i n P / P) f_0(w_{M,0} n_0 (M \cap P_0) / (M \cap P_0))$$

for $n \in \overline{N}$ and $n_0 \in M \cap \overline{N}_0$. (Notice that $w_{M,0}(M \cap \overline{N}_0) = (M \cap N_0)w_{M,0}$.) Then we have

$$\begin{aligned} I_{f,u'}(\varphi_\lambda) &= \int_{w_i w_{M,0} \overline{N}_0 (w_i w_{M,0})^{-1} \cap N_0} u'_0(\tilde{\varphi}(n w_i w_{M,0})) \tilde{f}(w_i w_{M,0} n P_0 / P_0) \eta(n)^{-1} dn. \end{aligned}$$

Hence, we may assume that P is minimal. By the same argument in (1), we may assume $f = 1$. If $f = 1$ then this integral is known as a Jacquet integral and the analytic continuation is well-known [Jac67].

We prove (2) and (4). By the same argument in (1), we may assume that $f = 1$. Take $w' \in W_{M_\eta}$ and $w'' \in W(M_\eta)^{-1}$ such that $w_i = w' w''$. Then we have $w_i \overline{N} w_i^{-1} \cap N_0 = (w' \overline{N}_0 (w')^{-1} \cap N_0) w' (w'' \overline{N}_0 (w'')^{-1} \cap N_0) (w')^{-1}$. The condition $w' \in W_{M_\eta}$ implies that $w'(\Sigma^+ \setminus \Sigma_\eta^+) = \Sigma^+ \setminus \Sigma_\eta^+$. Hence, $\text{supp } \eta \cap w' \Sigma^+ = \text{supp } \eta \cap w' \Sigma_\eta^+$. This implies

$$\begin{aligned} \text{supp } \eta \cap w' (w'' \Sigma^- \cap \Sigma^+) &= \text{supp } \eta \cap w_i \Sigma^- \cap w' \Sigma^+ \\ &= \text{supp } \eta \cap w_i \Sigma^- \cap w_i (w'')^{-1} \Sigma_\eta^+ \subset \text{supp } \eta \cap w_i \Sigma^- \cap w_i \Sigma^+ = \emptyset, \end{aligned}$$

i.e., η is trivial on $w'(w''\overline{N_0}(w'')^{-1} \cap N_0)(w')^{-1}$. Hence, we have

$$I_{1,w'}(\varphi) = \int_{w'\overline{N_0}(w')^{-1} \cap N_0} \int_{w''\overline{N_0}(w'')^{-1} \cap N_0} u'(\varphi(n_1 w' n_2 w'')) \eta(n_1)^{-1} dn_2 dn_1.$$

Put $P' = (w''P(w'')^{-1} \cap M_\eta)N_\eta$. By the definition of $W(M_\eta)$, we have $w''N_0(w'')^{-1} \supset N_0 \cap M_\eta$, this implies that P' (resp. $w''P(w'')^{-1} \cap M_\eta$) is a parabolic subgroup of G (resp. M_η). Define a G -module homomorphism $A(\sigma, \lambda): I(\sigma, \lambda) \rightarrow \text{Ind}_{P'}^G(w''(\sigma) \otimes e^{w''\lambda+\rho})$ by

$$(A(\sigma, \lambda)\varphi)(x) = \int_{w''\overline{N_0}(w'')^{-1} \cap N_0} \varphi(xnw'')dn.$$

By a result of Knapp and Stein [KS80], this homomorphism has a meromorphic continuation. We have

$$I_{1,w'}(\varphi) = \int_{w'\overline{N_0}(w')^{-1} \cap N_0} u'((A(\sigma, \lambda)\varphi)(nw'))\eta(n)^{-1}dn.$$

Notice that $w'\overline{N_0}(w')^{-1} \cap N_0 \subset M_\eta$. Hence we get (2) by (3).

To prove (4), we calculate $(w'')^{-1}\Sigma^- \cap \Sigma^+$. Since $(w'')^{-1} \in W(M_\eta)$, we have $(w'')^{-1}\Sigma_\eta^- \subset \Sigma^-$. Hence $(w'')^{-1}\Sigma_\eta^- \cap \Sigma^+ = \emptyset$. Then

$$\begin{aligned} (w'')^{-1}\Sigma^- \cap \Sigma^+ &= (w'')^{-1}(\Sigma^- \setminus \Sigma_\eta^-) \cap \Sigma^+ \\ &= (w'')^{-1}(w')^{-1}(\Sigma^- \setminus \Sigma_\eta^-) \cap \Sigma^+ \\ &= w_i^{-1}(\Sigma^- \setminus \Sigma_\eta^-) \cap \Sigma^+ \\ &= \Sigma^+ \setminus w_i^{-1}(\Sigma^+ \cup \Sigma_\eta^-). \end{aligned}$$

Hence we have $2\langle \alpha, \lambda + \nu \rangle / |\alpha|^2 \notin \mathbb{Z}_{\geq 0}$ for all $\alpha \in (w'')^{-1}\Sigma^- \cap \Sigma^+$. By an argument of Knapp and Stein [KS80], $A(\sigma, \mu)$ is holomorphic at $\mu = \lambda$ if λ satisfies the conditions of (4). Hence we get (4). \square

In the rest of this section, we denote the Bruhat filtration $I_i \subset J'(I(\sigma, \lambda))$ by $I_i(\lambda)$. The following result is a corollary of Proposition 4.4.

Lemma 4.5. *Let $x \in I'_i$. Then there exists a distribution $x_t \in I_i(\lambda + t\rho)$ with meromorphic parameter t such that $x_t|_{U_i}$ is a distribution with holomorphic parameter t and $(x|_{U_i})|_{t=0} = x$.*

Moreover, if $Tx = 0$ for $T \in U(\mathfrak{g})$, then $Tx_t = 0$.

Let $C^\infty(K, \sigma)$ be the space of σ -valued C^∞ -functions. For $X \in \mathfrak{g}$ and $\lambda \in \mathfrak{a}^*$, we define an operator $D(X, \lambda)$ on $C^\infty(K, \sigma)$ as follows. For $\varphi \in C^\infty(K, \sigma)$,

$$(D(X, \lambda)\varphi)(k) = \frac{d}{dt} (\sigma \otimes e^{\lambda+\rho})(\exp(-H(\exp(-tX)k)))\varphi(\kappa(\exp(-tX)k)) \Big|_{t=0}.$$

If we regard $I(\sigma, \lambda)$ as a subspace of $C^\infty(K, \sigma)$, $(X\varphi)(k) = (D(X, \lambda)\varphi)(k)$ for $\varphi \in I(\sigma, \lambda)$. It is easy to see that for some D_1 and D_2 we have $D(X, \lambda+t\rho) = D_1 + tD_2$ for all $t \in \mathbb{C}$.

Lemma 4.6. *Assume that the conditions of Lemma 3.6 (1)–(3) hold. For $x \in I'_i$ there exists a distribution $x_t \in I_i(\lambda + t\rho)$ with holomorphic parameter t defined near $t = 0$ such that $x_0 = x$ on U_i .*

PROOF. First we remark that $\eta = \eta'$ in Lemma 3.1 by the condition (2) of Lemma 3.6.

We prove by induction on i . If $i = 1$, then $x \in I'_1$. Take a distribution $x_t \in I_1(\lambda + t\rho)$ as in Lemma 4.5. Then $x_t|_{U_1}$ is holomorphic with respect to the parameter t . Since $\text{supp } x_t \subset X_1$, $x_t|_{(G/P) \setminus X_1}$ is holomorphic with respect to the parameter t . Hence x_t is holomorphic with respect to the parameter t on $U_1 \cup ((G/P) \setminus X_1) = G/P$. We have the lemma.

Assume that $i > 1$. First we prove the following claim: for $y \in I_{i-1}$, there exists a distribution $y_t \in I_{i-1}(\lambda + t\rho)$ with holomorphic parameter t defined near $t = 0$ such that $y_0 = y$. Using inductive hypothesis to $y|_{U_{i-1}}$, there exists a distribution $y_t^{(i-1)} \in I_{i-1}(\lambda + t\rho)$ with holomorphic parameter t defined near $t = 0$ such that $y_0^{(i-1)} = y$ on U_{i-1} . Since the supports of both sides are contained in $\bigcup_{j \leq i-1} N_0 w_j P/P$, we have $y_0^{(i-1)} = y$ on $\bigcup_{j \geq i-1} N_0 w_j P/P$. Using inductive hypothesis to $(y - y_0^{(i-1)})|_{U_{i-2}}$, there exists a distribution $y_t^{(i-2)} \in I_{i-2}(\lambda + t\rho)$ with holomorphic parameter t defined near $t = 0$ such that $y_0^{(i-2)} = y - y_0^{(i-1)}$ on U_{i-2} . Since the supports of both sides are contained in $\bigcup_{j \leq i-2} N_0 w_j P/P$, we have $y_0^{(i-1)} + y_0^{(i-2)} = y$ on $\bigcup_{j \geq i-2} N_0 w_j P/P$. Iterating this argument, for $j = 1, \dots, i-1$ there exists a distribution $y_t^{(j)} \in I_j(\lambda + t\rho)$ with holomorphic parameter t defined near $t = 0$ such that $y = y_0^{(1)} + \dots + y_0^{(i-1)}$. Hence we get the claim.

Now we prove the lemma. By Lemma 4.5, there exists a distribution $x'_t \in I_i(\lambda + t\rho)$ with meromorphic parameter t such that $x'_t|_{U_i}$ is holomorphic and $(x'_t|_{U_i})|_{t=0} = x$. Let $x'_t = \sum_{s=-p}^{\infty} x^{(s)} t^s$ be the Laurent series of x'_t . Now we

prove the following claim: if there exists a distribution $x'_t = \sum_{s=-p}^{\infty} x^{(s)} t^s \in I_i(\lambda + t\rho)$ with meromorphic parameter t defined near $t = 0$ such that $x'_t|_{U_i}$ is holomorphic and $(x'_t|_{U_i})|_{t=0} = x$, then there exists $x_t \in I(\lambda)$ with holomorphic parameter t defined near $t = 0$ such that $x_0|_{U_i} = x$. We prove the claim by induction on p .

If $p = 0$, we have nothing to prove. Assume $p > 0$. Take $E \in \mathfrak{n}_0$ and define differential operators E_0 and E_1 by $D(E, \lambda + t\rho) = E_0 + tE_1$. By Lemma 3.1, there exists a positive integer k such that $(E_0 + tE_1 - \eta(E))^k x'_t = 0$. Hence, we have $(E_0 - \eta(E))^k x^{(-p)} = 0$. Since $x_t|_{U_i}$ is holomorphic, we have $\text{supp } x^{(-p)} \subset \bigcup_{j < i} N_0 w_j P/P$. Hence we have $x^{(-p)} \in I_{i-1}$. By the claim stated in the third paragraph of this proof, there exists $x''_t \in I_{i-1}(\lambda + t\rho)$ with holomorphic parameter t defined near $t = 0$ such that $x''_0 = x^{(-p)}$. Using inductive hypothesis for $x'_t - t^{-p} x''_t$, we get the claim and the claim implies the lemma. \square

Theorem 4.7. (1) *The module I_i/I_{i-1} is non-zero if and only if the conditions of Lemma 3.6 (1)–(3) hold.*

(2) *If $I_i/I_{i-1} \neq 0$ then we have $I_i/I_{i-1} \simeq I'_i$.*

PROOF. Assume that the conditions of Lemma 3.6 (1)–(3) hold. We prove that the restriction map $\text{Res}_i: I_i \rightarrow I'_i$ is surjective.

For $x \in I'_i$, take $x_t \in I_i(\lambda + t\rho)$ as in Lemma 4.6. Then we have $\text{Res}_i(x_0) = (x_0)|_{U_i} = x$. Hence Res_i is surjective. \square

§5. Twisting functors

Arkhipov defined the *twisting functor* for $\tilde{w} \in \widetilde{W}$ [Ark04]. In this section, we define a modification of the twisting functor.

Let $\mathfrak{g}_\alpha^{\mathfrak{h}}$ be the root space of $\alpha \in \Delta$. Set $\mathfrak{u}_0 = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha^{\mathfrak{h}}$, $\overline{\mathfrak{u}_0} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}^{\mathfrak{h}}$ and $\mathfrak{u}_{0,\tilde{w}} = \text{Ad}(\tilde{w})\overline{\mathfrak{u}_0} \cap \mathfrak{u}_0$. Let ψ be a character of $\mathfrak{u}_{0,\tilde{w}}$. Put $S_{\tilde{w},\psi} = U(\mathfrak{g}) \otimes_{U(\mathfrak{u}_{0,\tilde{w}})} ((U(\mathfrak{u}_{0,\tilde{w}})^*)_{\mathfrak{h}\text{-finite}} \otimes_{\mathbb{C}} \psi)$. This is a right $U(\mathfrak{u}_{0,\tilde{w}})$ -module and left $U(\mathfrak{g})$ -module. We define a $U(\mathfrak{g})$ -bimodule structure on $S_{\tilde{w},\psi}$ in the following way. Let $\{e_1, \dots, e_l\}$ be a basis of $\mathfrak{u}_{0,\tilde{w}}$ such that each e_i is a root vector and $\bigoplus_{s \leq t-1} \mathbb{C}e_s$ is an ideal of $\bigoplus_{s \leq t} \mathbb{C}e_s$ for each $t = 1, 2, \dots, l$. Notice that a multiplicative set $\{(e_k - \psi(e_k))^n \mid n \in \mathbb{Z}_{\geq 0}\}$ satisfies the Ore condition for $k = 1, 2, \dots, l$. Then we can consider the localization of $U(\mathfrak{g})$ by $\{(e_k - \psi(e_k))^n \mid n \in \mathbb{Z}_{\geq 0}\}$. We denote the resulting algebra by $U(\mathfrak{g})_{e_k - \psi(e_k)}$. Put $S_{e_k - \psi(e_k)} = U(\mathfrak{g})_{e_k - \psi(e_k)} / U(\mathfrak{g})$. Then $S_{e_k - \psi(e_k)}$ is a $U(\mathfrak{g})$ -bimodule.

Proposition 5.1. *As a right $U(\mathfrak{u}_{0,w})$ -module and left $U(\mathfrak{g})$ -module, we have $S_{\tilde{w},\psi} \simeq S_{e_1-\psi(e_1)} \otimes_{U(\mathfrak{g})} S_{e_2-\psi(e_2)} \otimes_{U(\mathfrak{g})} \cdots \otimes_{U(\mathfrak{g})} S_{e_l-\psi(e_l)}$. Moreover, the $U(\mathfrak{g})$ -bimodule structure induced from this isomorphism is independent of a choice of e_i .*

The proof of this proposition is similar to that of Arkhipov [Ark04, Theorem 2.1.6]. We omit it. An element of the right hand side is written as a sum of a form $(e_1 - \eta(e_1))^{-(k_1+1)} \otimes \cdots \otimes (e_l - \eta(e_l))^{-(k_l+1)} T$ for $T \in U(\mathfrak{g})$. We denote this element by $(e_1 - \eta(e_1))^{-(k_1+1)} \cdots (e_l - \eta(e_l))^{-(k_l+1)} T$ for short.

Proposition 5.1 gives the $U(\mathfrak{g})$ -bimodule structure of $S_{\tilde{w},\psi}$. For a $U(\mathfrak{g})$ -module V , we define a $U(\mathfrak{g})$ -module $T_{\tilde{w},\psi} V$ by $T_{\tilde{w},\psi} V = S_{\tilde{w},\psi} \otimes_{U(\mathfrak{g})} (\tilde{w}V)$. (Recall that $\tilde{w}V$ is a \mathfrak{g} -module twisted by \tilde{w} . See Notation.) This gives the twisting functor $T_{\tilde{w},\psi}$. If ψ is the trivial representation, $T_{\tilde{w},\psi}$ is the twisting functor defined by Arkhipov. We put $T_{\tilde{w}} = T_{\tilde{w},0}$ where 0 is the trivial representation.

The restriction map gives a map $N_K(\mathfrak{h})/Z_K(\mathfrak{h}) \rightarrow W$ and its kernel is isomorphic to $N_{M_0}(\mathfrak{t}_0)/Z_{M_0}(\mathfrak{t}_0)$ (Recall that \mathfrak{t}_0 is a Cartan subalgebra of \mathfrak{m}_0). The last group is isomorphic to \widetilde{W}_{M_0} .

Lemma 5.2. *Let $w \in W$. Then there exists $\iota(w) \in N_K(\mathfrak{h})$ such that $\text{Ad}(\iota(w))|_{\mathfrak{a}_0} = w$ and $\text{Ad}(\iota(w))(\Delta_{M_0}^+) = \Delta_{M_0}^+$.*

PROOF. Since $W \simeq N_K(\mathfrak{a}_0)/Z_K(\mathfrak{a}_0)$, there exists $k \in N_K(\mathfrak{a}_0)$ such that $\text{Ad}(k)|_{\mathfrak{a}_0} = w$. Then k normalizes M_0 . Hence, there exists $m \in M_0$ such that km normalizes T_0 . This implies $km \in N_K(A_0 T_0)$. Take $w' \in N_{M_0}(\mathfrak{t}_0)$ such that $\text{Ad}(kmw')(\Delta_{M_0}^+) = \Delta_{M_0}^+$ and put $\iota(w) = kmw'$. Then $\iota(w)$ satisfies the conditions of the lemma. \square

The map ι gives an injective map $W \rightarrow N_K(\mathfrak{h})/Z_K(\mathfrak{h})$. Since the group $N_K(\mathfrak{h})/Z_K(\mathfrak{h})$ can be regarded as a subgroup of \widetilde{W} , we can regard W as a subgroup of \widetilde{W} . Hence, we can define the twisting functor $T_{w,\psi}$ for $w \in W$ and the character ψ of $\text{Ad}(w)\overline{\mathfrak{n}_0} \cap \mathfrak{n}_0$. For a simplicity, we write w instead of $\iota(w)$. (We regard W as a subgroup of \widetilde{W} by ι .)

Proposition 5.3. *Let $w, w' \in W$ and ψ a character of $\text{Ad}(ww')\overline{\mathfrak{n}_0} \cap \mathfrak{n}_0$. Assume that $\ell(w) + \ell(w') = \ell(ww')$ where $\ell(w)$ is the length of $w \in W$. Then we have $T_{w,\psi} T_{w',w^{-1}\psi} = T_{ww',\psi}$.*

PROOF. By the assumption, we have $\Sigma^+ \cap ww'\Sigma^- = (\Sigma^+ \cap w\Sigma^-) \cup w(\Sigma^+ \cap w'\Sigma^-)$. Put $\Delta_0^\pm = \Delta^\pm \setminus \Delta_{M_0}^\pm$. Then we have $\Delta_0^+ \cap ww'\Delta_0^- = (\Delta_0^+ \cap w\Delta_0^-) \cup w(\Delta_0^+ \cap w'\Delta_0^-)$. Since $w\Delta_{M_0}^\pm = \Delta_{M_0}^\pm$, we have $\Delta_0^+ \cap w\Delta_0^- = \Delta^+ \cap w\Delta^-$. Hence, $\Delta^+ \cap ww'\Delta^- = (\Delta^+ \cap w\Delta^-) \cup w(\Delta^+ \cap w'\Delta^-)$. This implies that

$\tilde{\ell}(w) + \tilde{\ell}(w') = \tilde{\ell}(ww')$ where $\tilde{\ell}(w)$ is the length of w as an element of \widetilde{W} . Hence, the proposition follows from the construction of the twisting functor (See Andersen and Lauritzen [AL03, Remark 6.1 (ii)]). \square

Lemma 5.4. *Let e be a nilpotent element of \mathfrak{g} , $X \in \mathfrak{g}$ and $k \in \mathbb{Z}_{\geq 0}$. For $c \in \mathbb{C}$ we have the following equation in $U(\mathfrak{g})_{e-c}$.*

$$X(e-c)^{-(k+1)} = \sum_{n=0}^{\infty} \binom{n+k}{k} (e-c)^{-(n+k+1)} \text{ad}(e)^n(X).$$

PROOF. We prove the lemma by induction on k . If $k = 0$, then the lemma is well-known. Assume that $k > 0$. Then we have

$$\begin{aligned} X(e-c)^{-(k+1)} &= \sum_{k_0=0}^{\infty} (e-c)^{-(k_0+1)} \text{ad}(e)^{k_0}(X)(e-c)^{-k} \\ &= \sum_{k_0=0}^{\infty} \sum_{k_1=0}^{\infty} \binom{k_1+k-1}{k-1} (e-c)^{-(k_0+k_1+k+1)} \text{ad}(e)^{k_0+k_1}(X) \\ &= \sum_{n=0}^{\infty} \sum_{l'=0}^n \binom{l'+k-1}{k-1} (e-c)^{-(n+k+1)} \text{ad}(e)^n(X) \\ &= \sum_{n=0}^{\infty} \binom{n+k}{k} (e-c)^{-(n+k+1)} \text{ad}(e)^n(X). \end{aligned}$$

This proves the lemma. \square

§6. The module I_i/I_{i-1}

Put $J_i = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$, where \mathfrak{n} acts $J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$ trivially. In this section, we prove the following theorem.

Theorem 6.1. *Assume that $I_i/I_{i-1} \neq 0$. Then we have $I_i/I_{i-1} \simeq T_{w_i,\eta}J_i$.*

Notice that $\mathfrak{u}_{0,w_i} = \text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0$ since $w_i(\Delta_M^+) \subset \Delta^+$. In this section fix $i \in \{1, \dots, l\}$ and a basis $\{e_1, e_2, \dots, e_l\}$ of \mathfrak{u}_{0,w_i} such that each vector e_i is a root vector and $\bigoplus_{s \leq t-1} \mathbb{C}e_s$ is an ideal of $\bigoplus_{s \leq t} \mathbb{C}e_s$. Let α_s be the restricted root with respect to e_s . As in Section 3, for $\mathbf{k} = (k_1, \dots, k_l) \in \mathbb{Z}_{\geq 0}^l$ we denote $\text{ad}(e_l)^{k_l} \dots \text{ad}(e_1)^{k_1}$ by $\text{ad}(e)^{\mathbf{k}}$ and $((-x_1)^{k_1}/k_1!) \dots ((-x_l)^{k_l}/k_l!)$ by $f_{\mathbf{k}}$.

Lemma 6.2. *We have*

$$I'_i = \left\{ \sum_{s=1}^t \delta_i(T_s, f_s \eta_i^{-1}, u'_s) \mid \begin{array}{l} T_s \in U(\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0), \quad f_s \in \mathcal{P}(O_i), \\ u'_s \in J'_{w_i^{-1}\eta}(\sigma \otimes (\lambda + \rho)) \end{array} \right\}.$$

PROOF. By Lemma 3.3, we have

$$T((f \otimes u')\delta_i) = \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^l} \delta_i(\text{ad}(e)^{\mathbf{k}}T, f f_{\mathbf{k}}, u')$$

for $T \in U(\mathfrak{g})$, $f \in \mathcal{P}(O_i)\eta_i^{-1}$ and $u' \in \sigma'$. Hence, the left hand side is a subset of the right hand side. Define $f'_{\mathbf{k}} \in \mathcal{P}(O_i)$ by $f'_{\mathbf{k}} = (x_1^{k_1}/k_1!) \cdots (x_l^{k_l}/k_l!)$. By the similar calculation of Lemma 3.3, we have

$$\delta_i(T, f, u') = \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^l} (\text{ad}(e)^{\mathbf{k}}T)((f f'_{\mathbf{k}}) \otimes u')\delta_i).$$

This implies that the right hand side is contained in the left hand side. \square

By the definition of the twisting functor and Poincaré-Birkhoff-Witt theorem, we have the following lemma. For $\mathbf{k} = (k_1, \dots, k_l) \in \mathbb{Z}^l$ put $(e - \eta(e))^{\mathbf{k}} = (e_1 - \eta(e_1))^{k_1} \cdots (e_l - \eta(e_l))^{k_l} \in S_{w, \eta}$. Set $\mathbf{1} = (1, \dots, 1) \in \mathbb{Z}^l$.

Lemma 6.3. *Let V be a \mathfrak{p} -module. Then we have a \mathbb{C} -vector space isomorphism*

$$\left(\sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^l} \mathbb{C}(e - \eta(e))^{-(\mathbf{k} + \mathbf{1})} \right) \otimes_{U(\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0)} U(\mathfrak{g}) \otimes_{U(\text{Ad}(w_i)\mathfrak{p})} w_i V \simeq T_{w_i}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V)$$

given by $E \otimes T \otimes v \mapsto ET \otimes (1 \otimes v)$. (Notice that $ET \in S_{w_i, 0}$.)

PROOF OF THEOREM 6.1. By Lemma 6.2, we have an isomorphism as a vector space,

$$I'_i \simeq \mathcal{P}(O_i) \otimes_{U(\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0)} U(\mathfrak{g}) \otimes_{U(\text{Ad}(w_i)\mathfrak{p})} w_i J'_{w_i^{-1}\eta}(\sigma \otimes (\lambda + \rho))$$

given by $\delta_i(T, f, u') \mapsto f \otimes T \otimes u'$.

Generalized Jacquet modules of parabolic induction

Notice that $\mathfrak{u}_{0,w_i} = \text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0$ since $w_i \in W(M)$. By Lemma 6.3, we have

$$T_{w_i,\eta}(J_i) \simeq \left(\sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^l} \mathbb{C}(e - \eta(e))^{-(\mathbf{k}+1)} \right) \otimes_{U(\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0)} U(\mathfrak{g}) \otimes_{U(\text{Ad}(w_i)\mathfrak{p})} w_i J'_{w_i^{-1}\eta}(\sigma \otimes (\lambda + \rho)).$$

Here we remark $\sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^l} \mathbb{C}(e - \eta(e))^{-(\mathbf{k}+1)}$ is an $\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0$ -stable subspace of $S_{w_i,\eta}$. Hence, we can define a \mathbb{C} -vector space isomorphism $\Phi: T_{w_i,\eta}(J_i) \rightarrow I'_i$ by

$$\Phi((e - \eta(e))^{-(\mathbf{k}+1)} \otimes T \otimes u') = \delta_i(T, f_{\mathbf{k}}\eta_i^{-1}, u').$$

We prove that Φ is a \mathfrak{g} -homomorphism.

Fix $X \in \mathfrak{g}$. We prove that

$$\Phi(X((e - \eta(e))^{-(\mathbf{k}+1)} \otimes T \otimes u')) = X\Phi((e - \eta(e))^{-(\mathbf{k}+1)} \otimes T \otimes u').$$

By Lemma 5.4, we have

$$\begin{aligned} & X((e - \eta(e))^{-(\mathbf{k}+1)} \otimes T \otimes u') \\ &= \sum_{p_s \geq 0} \binom{p_1 + k_1}{k_1} \cdots \binom{p_l + k_l}{k_l} (e - \eta(e))^{-(\mathbf{k}+\mathbf{p}+1)} \otimes (\text{ad}(e)^{\mathbf{p}}X)T \otimes u'. \end{aligned}$$

where $\mathbf{p} = (p_1, \dots, p_l)$. Hence, we have

$$\begin{aligned} & \Phi(X((e - \eta(e))^{-(\mathbf{k}+1)} \otimes T \otimes u')) \\ &= \sum_{p_s \geq 0} \delta_i \left((\text{ad}(e)^{\mathbf{p}}X)T, \left(\frac{(-x_1)^{k_1+p_1}}{k_1!p_1!} \cdots \frac{(-x_l)^{k_l+p_l}}{k_l!p_l!} \right) \eta_i^{-1}, u' \right). \end{aligned}$$

By Lemma 3.3, we have

$$\begin{aligned} X\Phi((e - \eta(e))^{-(\mathbf{k}+1)} \otimes T \otimes u') &= X\delta_i(T, f_{\mathbf{k}}\eta_i^{-1}, u') \\ &= \sum_{\mathbf{p} \in \mathbb{Z}_{\geq 0}^l} \delta_i((\text{ad}(e)^{\mathbf{p}}X)T, f_{\mathbf{k}}f_{\mathbf{p}}\eta_i^{-1}, u'). \end{aligned}$$

Hence, we have the theorem. \square

§7. The module $J_\eta^*(I(\sigma, \lambda))$

Now we investigate a module $J_\eta^*(I(\sigma, \lambda))$. For a finite-length Fréchet representation V of G , put $J(V) = (\varprojlim_{k \rightarrow \infty} (V_{K\text{-finite}}/\mathfrak{n}_0^k V_{K\text{-finite}}))_{\mathfrak{a}\text{-finite}}$. This is also called the Jacquet module of V [Cas80]. Define a category \mathcal{O}'_{P_0} by the full subcategory of finitely generated \mathfrak{g} -modules consisting an object V satisfying the following conditions.

- (1) The action of \mathfrak{p}_0 is locally finite. (In particular, the action of \mathfrak{n}_0 is locally nilpotent.)
- (2) The module V is $Z(\mathfrak{g})$ -finite.
- (3) The group M_0 acts on V and its differential coincides with the action of $\mathfrak{m}_0 \subset \mathfrak{g}$.
- (4) For $\nu \in \mathfrak{a}_0^*$ let V_ν be the generalized \mathfrak{a}_0 -weight space with weight ν . Then $V = \bigoplus_{\nu \in \mathfrak{a}_0^*} V_\nu$ and $\dim V_\nu < \infty$.

We define the category $\mathcal{O}'_{\overline{P}_0}$ similarly. Then for a finite-length Fréchet representation V of G we have $J(V) \in \mathcal{O}'_{\overline{P}_0}$ and $J^*(V) \in \mathcal{O}'_{P_0}$. For a $U(\mathfrak{g})$ -module V , put $D'(V) = (V^*)_{\mathfrak{h}\text{-finite}}$ and $C(V) = (D'(V))^*$. Denote a full-subcategory of \mathfrak{g} -modules consisting finitely-generated and locally $\mathfrak{h} \oplus \mathfrak{u}$ -finite modules by \mathcal{O}' . If V is an object of the category \mathcal{O}' then $D'D'(V) \simeq V$. The relation between J^* and J is as follows.

Proposition 7.1. *Let V be a finite-length Fréchet representation of G . Then we have $J^*(V) \simeq D'(J(V))$.*

The character $\eta: \mathfrak{n}_0 \rightarrow \mathbb{C}$ defines an algebra homomorphism $U(\mathfrak{n}_0) \rightarrow \mathbb{C}$ by the universality of the universal enveloping algebra. Let $\text{Ker } \eta$ be the kernel of this algebra homomorphism and put $\Gamma_\eta(V) = \{v \in V \mid \text{for some } k, (\text{Ker } \eta)^k v = 0\}$. First we prove the following proposition.

Proposition 7.2. *Let V be a finite-length Fréchet representation of G . Then we have $J_\eta^*(V) \simeq \Gamma_\eta(J(V)^*)$.*

PROOF. Recall that $\mathfrak{p}_\eta = \mathfrak{m}_\eta \oplus \mathfrak{a}_\eta \oplus \mathfrak{n}_\eta$ is the complexification of the Lie algebra of the parabolic subgroup corresponding to $\text{supp } \eta$ (Section 4). If $\text{supp } \eta = \Pi$, this proposition is proved by Matumoto [Mat90, Theorem 4.9.2].

Put $I = V_{K\text{-finite}}$. Let $\eta_0: U(\mathfrak{m} \cap \mathfrak{n}_0) \rightarrow \mathbb{C}$ be the restriction of η on $U(\mathfrak{m} \cap \mathfrak{n}_0)$. Then we have

$$J_\eta^*(V) = \varprojlim_{k,l} (I/\mathfrak{n}_\eta^l (\text{Ker } \eta_0)^k I)^* = \varprojlim_{k,l} ((I/\mathfrak{n}_\eta^l I)/(\text{Ker } \eta_0)^k (I/\mathfrak{n}_\eta^l I))^*.$$

For a $U(\mathfrak{g})$ -module V_0 , put $G(V_0) = (\varprojlim_k V_0/\mathfrak{n}_0^k V_0)_{\mathfrak{a}\text{-finite}}$. For a $U(\mathfrak{m}_\eta \oplus \mathfrak{a}_\eta)$ -module V_1 , put $G_{M_\eta}(V_1) = (\varprojlim_k V_1/(\mathfrak{m}_\eta \cap \mathfrak{n}_0)^k V_1)_{(\mathfrak{m} \cap \mathfrak{a}_0)\text{-finite}}$. Since $I/\mathfrak{n}_\eta^l I$ is a Harish-Chandra module of $\mathfrak{m}_\eta \oplus \mathfrak{a}_\eta$, $J_{\eta_0}^*(I/\mathfrak{n}_\eta^l I) = \Gamma_{\eta_0}(G_{M_\eta}(I/\mathfrak{n}_\eta^l I)^*)$ by the result of Matumoto. Taking a subspace annihilated by $(\text{Ker } \eta_0)^k$, we have

$$((I/\mathfrak{n}_\eta^l I)/(\text{Ker } \eta_0)^k(I/\mathfrak{n}_\eta^l I))^* = (G_{M_\eta}(I/\mathfrak{n}_\eta^l I)/(\text{Ker } \eta_0)^k G_{M_\eta}(I/\mathfrak{n}_\eta^l I))^*.$$

Since I is a finitely-generated $U(\mathfrak{n}_0)$ -module, the left hand side is finite-dimensional. Hence, we have

$$(I/\mathfrak{n}_\eta^l I)/(\text{Ker } \eta_0)^k(I/\mathfrak{n}_\eta^l I) = G_{M_\eta}(I/\mathfrak{n}_\eta^l I)/(\text{Ker } \eta_0)^k G_{M_\eta}(I/\mathfrak{n}_\eta^l I).$$

It is sufficient to prove that $G_{M_\eta}(I/\mathfrak{n}_\eta^l I) = G(I)/\mathfrak{n}_\eta^l G(I)$. We have

$$(I/\mathfrak{n}_\eta^l I)/(\mathfrak{m}_\eta \cap \mathfrak{n}_0)^k(I/\mathfrak{n}_\eta^l I) = I/(\mathfrak{m}_\eta \cap \mathfrak{n}_0)^k \mathfrak{n}_\eta^l I = G(I)/(\mathfrak{m}_\eta \cap \mathfrak{n}_0)^k \mathfrak{n}_\eta^l G(I).$$

Taking the projective limit we have $G_{M_\eta}(I/\mathfrak{n}_\eta^l I) = G_{M_\eta}(G(I)/\mathfrak{n}_\eta^l G(I))$. Since $G(I)/\mathfrak{n}_\eta^l G(I) \in \mathcal{O}'_{M_\eta \cap \overline{P}_0}$ we have $G_{M_\eta}(G(I)/\mathfrak{n}_\eta^l G(I)) = G(I)/\mathfrak{n}_\eta^l G(I)$. \square

Combining Theorem 6.1, Proposition 7.2 and the automatic continuation theorem [Wal83, Theorem 4.8], we have the following theorem.

Theorem 7.3. *There exists a filtration $0 = \tilde{I}_1 \subset \cdots \subset \tilde{I}_r = J_\eta^*(I(\sigma, \lambda))$ such that $\tilde{I}_i/\tilde{I}_{i-1} \simeq \Gamma_\eta(C(T_{w_i}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} J^*(\sigma \otimes e^{\lambda+\rho})))$.*

§8. Whittaker vectors

In this section we study Whittaker vectors of $I(\sigma, \lambda)'$ and $(I(\sigma, \lambda)_{K\text{-finite}})^*$ (Definition 3.8).

For i such that $I_i/I_{i-1} \neq 0$, we define some maps as follows. Let γ_1 be the first projection with respect to the decomposition $U(\mathfrak{g}) = U(\mathfrak{l}_\eta) \oplus (\overline{\mathfrak{n}}_\eta U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_\eta)$. Notice that by Lemma 3.6 if $I_i/I_{i-1} \neq 0$ then we have $\mathfrak{l}_\eta \cap \text{Ad}(w_i)\overline{\mathfrak{n}} \subset \mathfrak{n}_0$. Define γ_2 by the first projection with respect to the decomposition $U(\mathfrak{l}_\eta) = U(\mathfrak{l}_\eta \cap \text{Ad}(w_i)\mathfrak{p}) \oplus U(\mathfrak{l}_\eta) \text{Ker } \eta|_{\mathfrak{l}_\eta \cap \text{Ad}(w_i)\overline{\mathfrak{n}}}$. Let γ_3 be the first projection with respect to the decomposition $U(\mathfrak{l}_\eta \cap \text{Ad}(w_i)\mathfrak{p}) = U(\mathfrak{l}_\eta \cap \text{Ad}(w_i)\mathfrak{l}) \oplus (\mathfrak{l}_\eta \cap \text{Ad}(w_i)\mathfrak{n})U(\mathfrak{l}_\eta \cap \text{Ad}(w_i)\mathfrak{p})$. Finally define γ_4 by the first projection with respect to the decomposition $U(\mathfrak{l}_\eta \cap \text{Ad}(w_i)\mathfrak{l}) = U(\mathfrak{h}) \oplus ((\overline{\mathfrak{u}}_0 \cap \mathfrak{l}_\eta \cap \text{Ad}(w_i)\mathfrak{l})U(\mathfrak{l}_\eta \cap \text{Ad}(w_i)\mathfrak{l}) + U(\mathfrak{l}_\eta \cap \text{Ad}(w_i)\mathfrak{l})(\mathfrak{l}_\eta \cap \text{Ad}(w_i)\mathfrak{l} \cap \mathfrak{u}_0))$. Then

the restriction of $\gamma_4 \circ \gamma_3 \circ \gamma_2 \circ \gamma_1$ on $Z(\mathfrak{g})$ is the (non-shifted) Harish-Chandra homomorphism. If $x \in \text{Wh}_\eta(I_i/I_{i-1})$ then $Tx = \gamma_2\gamma_1(T)x$ for $T \in Z(\mathfrak{g})$.

$$\begin{aligned}\gamma_1: U(\mathfrak{g}) &= U(\mathfrak{l}_\eta) \oplus (\overline{\mathfrak{n}_\eta}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_\eta) \rightarrow U(\mathfrak{l}_\eta), \\ \gamma_2: U(\mathfrak{l}_\eta) &= U(\mathfrak{l}_\eta \cap \text{Ad}(w_i)\mathfrak{p}) \oplus U(\mathfrak{l}_\eta) \text{Ker } \eta|_{\mathfrak{l}_\eta \cap \text{Ad}(w_i)\overline{\mathfrak{n}}} \rightarrow U(\mathfrak{l}_\eta \cap \text{Ad}(w_i)\mathfrak{p}), \\ \gamma_3: U(\mathfrak{l}_\eta \cap \text{Ad}(w_i)\mathfrak{p}) &= U(\mathfrak{l}_\eta \cap \text{Ad}(w_i)\mathfrak{l}) \oplus (\mathfrak{l}_\eta \cap \text{Ad}(w_i)\mathfrak{n})U(\mathfrak{l}_\eta \cap \text{Ad}(w_i)\mathfrak{p}), \\ &\rightarrow U(\mathfrak{l}_\eta \cap \text{Ad}(w_i)\mathfrak{l}) \\ \gamma_4: U(\mathfrak{l}_\eta \cap \text{Ad}(w_i)\mathfrak{l}) &= U(\mathfrak{h}) \oplus ((\overline{\mathfrak{u}_0} \cap \mathfrak{l}_\eta \cap \text{Ad}(w_i)\mathfrak{l})U(\mathfrak{l}_\eta \cap \text{Ad}(w_i)\mathfrak{l}) \\ &\quad + U(\mathfrak{l}_\eta \cap \text{Ad}(w_i)\mathfrak{l})(\mathfrak{l}_\eta \cap \text{Ad}(w_i)\mathfrak{l} \cap \mathfrak{u}_0)) \rightarrow U(\mathfrak{h}).\end{aligned}$$

Lemma 8.1. *Let V be a $U(\mathfrak{g})$ -module with an infinitesimal character $\tilde{\lambda}$, χ a character of $Z(\mathfrak{g})$ such that $z \in Z(\mathfrak{g})$ acts by $\chi(z)$ on V . Take a nonzero element $v \in V$ such that $(\gamma_3\gamma_2\gamma_1(z) - \chi(z))v = 0$. Moreover, assume that there exists $\mu \in \mathfrak{a}^*$ such that $Hv = (w_i\mu + \rho_0)(H)v$ for all $H \in \text{Ad}(w_i)\mathfrak{a}$. Then there exists $\tilde{w} \in \tilde{W}$ such that $\tilde{w}\tilde{\lambda}|_{\mathfrak{a}} = \mu$.*

PROOF. Put $Z = (\gamma_3\gamma_2\gamma_1(Z(\mathfrak{g}))U(\text{Ad}(w_i)\mathfrak{a}))$. By the assumption, there exists a character χ_0 of Z such that $zv = \chi_0(z)v$ for all $z \in Z$. By a theorem of Harsh-Chandra, $\tilde{\gamma}_4|_Z$ is injective and $\tilde{\gamma}_4(Z) \subset U(\mathfrak{h})$ is finite. Hence there exists an element $\tilde{\lambda}_1 \in \mathfrak{h}^*$ such that $\tilde{\lambda}_1 \circ \gamma_4 = \chi_0$ where we denote the algebra homomorphism $U(\mathfrak{h}) \rightarrow \mathbb{C}$ induced from $\tilde{\lambda}_1$ by the same letter $\tilde{\lambda}_1$. Since V has an infinitesimal character $\tilde{\lambda}$, we have $\tilde{\lambda}_1 \in \tilde{W}\tilde{\lambda} + \tilde{\rho}$. Since γ_4 is trivial on $U(\text{Ad}(w_i)\mathfrak{a})$, $\tilde{\lambda}_1|_{\text{Ad}(w_i)\mathfrak{a}} = (w_i\mu + \rho_0)|_{\text{Ad}(w_i)\mathfrak{a}}$. The restriction of $\tilde{\rho}$ to \mathfrak{a}_0 is ρ_0 . Hence $\tilde{\rho}|_{\text{Ad}(w_i)\mathfrak{a}} = \rho_0|_{\text{Ad}(w_i)\mathfrak{a}}$. Then for some $\tilde{w} \in \tilde{W}$ we have $w_i\mu|_{\text{Ad}(w_i)\mathfrak{a}} = \tilde{w}\tilde{\lambda}|_{\text{Ad}(w_i)\mathfrak{a}}$. We get the lemma. \square

Lemma 8.2. *Let $X_1, \dots, X_n \in \mathfrak{g}$, $f_1 \in C^\infty(O_i)$, $f_2 \in C^\infty(U_i)$, $u' \in (\sigma \otimes e^{\lambda+\rho})'$. Assume that $\tilde{R}'_i(X_s)(f_2) = 0$ for all $s = 1, \dots, n$. Then we have*

$$\delta_i(X_1 \cdots X_n, f_1 f_2, u') = \delta_i(X_1 \cdots X_n, f_1, u') f_2.$$

PROOF. Put $T = X_1 \cdots X_n$. By the assumption and Leibniz's rule, we have

$$f_2(nw_i)(\tilde{R}_i(T)\varphi)(nw_i) = (\tilde{R}_i(T)(\varphi f_2))(nw_i)$$

Hence, by the definition, for $\varphi \in C_c^\infty(U_i, \mathcal{L})$, we have

$$\begin{aligned}
 & \langle \delta_i(T, f_1 f_2, u'), \varphi \rangle \\
 &= \int_{w_i \bar{N} w_i^{-1} \cap N_0} f_1(nw_i) f_2(nw_i) (u'(\tilde{R}_i(T)\varphi)(nw_i)) dn \\
 &= \int_{w_i \bar{N} w_i^{-1} \cap N_0} f_1(nw_i) (u'(\tilde{R}_i(T)(\varphi f_2))(nw_i)) dn \\
 &= \langle \delta_i(T, f_1, u'), f_2 \varphi \rangle \\
 &= \langle \delta_i(T, f_1, u') f_2, \varphi \rangle.
 \end{aligned}$$

We get the lemma. \square

Lemma 8.3. *For $\nu \in \mathfrak{a}^*$ put*

$$V(\nu) = \left\{ \sum_s \delta_i(S_s, h_s \eta_i^{-1}, v'_s) \left| \begin{array}{l} S_s \in U(\text{Ad}(w_i) \bar{\mathfrak{n}} \cap \bar{\mathfrak{n}}_0), \ h_s \in \mathcal{P}(O_i), \\ v'_s \in J'_{w_i^{-1} \eta}(\sigma \otimes e^{\lambda+\rho}), \\ w_i^{-1}(\text{wt } h_s + \text{wt } S_s)|_{\mathfrak{a}} = \nu \end{array} \right. \right\}.$$

Here, $\text{wt } h_s$ is an \mathfrak{a}_0 -weight of h_s with respect to D_i (see page 11) and $\text{wt } S_s$ is an \mathfrak{a}_0 -weight of S_s with respect to the adjoint action. Define $\tilde{\eta}_i \in C^\infty(U_i)$ by $\tilde{\eta}_i(nn_0 w_i P/P) = \eta_i(n)$ for $n \in w_i \bar{N} w_i^{-1} \cap N_0$ and $n_0 \in w_i \bar{N} w_i^{-1} \cap \bar{N}_0$.

(1) Let $X \in U(\mathfrak{l}_\eta \cap \text{Ad}(w_i) \mathfrak{p})$. Assume that X is an \mathfrak{a}_0 -weight vector. For $\delta_i(T, f \eta_i^{-1}, u') \in V(\nu)$, we have

$$X \delta_i(T, f \eta_i^{-1}, u') - (X \delta_i(T, f \eta_i^{-1}, u')) \tilde{\eta}_i^{-1} \in \sum_{\nu' > \nu} V(\nu' + w_i^{-1} \text{wt } T|_{\mathfrak{a}}).$$

here, $\text{wt } T$ is an \mathfrak{a}_0 -weight of T with respect to the adjoint action.

(2) For $\delta_i(S_s, h_s \eta_i^{-1}, v'_s) \in V(\nu)$, we have

$$\sum_s \delta_i(S_s, h_s, v'_s) \tilde{\eta}_i^{-1} \notin \sum_{\nu' > \nu} V(\nu').$$

PROOF. (1) Fix a basis $\{e_1, e_2, \dots, e_l\}$ of \mathfrak{u}_{0, w_i} such that each vector e_i is a root vector and $\bigoplus_{s \leq t-1} \mathbb{C} e_s$ is an ideal of $\bigoplus_{s \leq t} \mathbb{C} e_s$. Let α_s be the restricted root of e_s . As in Section 3, for $\mathbf{k} = (k_1, \dots, k_l) \in \mathbb{Z}_{\geq 0}^l$ we denote $\text{ad}(e_l)^{k_l} \dots \text{ad}(e_1)^{k_1}$ by $\text{ad}(e)^{\mathbf{k}}$ and $((-x_1)^{k_1}/k_1!) \dots ((-x_l)^{k_l}/k_l!)$ by $f_{\mathbf{k}}$. By Lemma 3.3,

$$X \delta_i(T, f \eta_i^{-1}, u') = \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^l} \delta_i((\text{ad}(e)^{\mathbf{k}} X) T, f f_{\mathbf{k}} \eta_i^{-1}, u').$$

Take $a_{\mathbf{k}}^{(p)} \in U(\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0)$, $b_{\mathbf{k}}^{(p)} \in U(\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \bar{\mathfrak{n}}_0)$ and $c_{\mathbf{k}}^{(p)} \in U(\text{Ad}(w_i)\mathfrak{p})$ such that $(\text{ad}(e)^{\mathbf{k}}X)T = \sum_p a_{\mathbf{k}}^{(p)} b_{\mathbf{k}}^{(p)} c_{\mathbf{k}}^{(p)}$ and $\text{wt}((\text{ad}(e)^{\mathbf{k}}X)T) = \text{wt } a_{\mathbf{k}}^{(p)} + \text{wt } b_{\mathbf{k}}^{(p)} + \text{wt } c_{\mathbf{k}}^{(p)}$. Then

$$\begin{aligned} & \delta_i((\text{ad}(e)^{\mathbf{k}}X)T, f f_{\mathbf{k}} \eta_i^{-1}, u') \\ &= \sum_p \delta_i(a_{\mathbf{k}}^{(p)} b_{\mathbf{k}}^{(p)} c_{\mathbf{k}}^{(p)}, f f_{\mathbf{k}} \eta_i^{-1}, u') \\ &= \sum_p \delta_i(b_{\mathbf{k}}^{(p)}, R'_i(-a_{\mathbf{k}}^{(p)})(f f_{\mathbf{k}} \eta_i^{-1}), \text{Ad}(w_i)^{-1}(c_{\mathbf{k}}^{(p)})u') \end{aligned}$$

By the Leibniz rule, there exists a subset $\mathcal{A}_{\mathbf{k}}^{(p)} \subset \{(a', a'') \in U(\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0)^2 \mid \text{wt } a' + \text{wt } a'' = \text{wt } a_{\mathbf{k}}^{(p)}, a'' \notin \mathbb{C}\}$ such that

$$\begin{aligned} & \delta_i(b_{\mathbf{k}}^{(p)}, R'_i(-a_{\mathbf{k}}^{(p)})(f f_{\mathbf{k}} \eta_i^{-1}) - R'_i(-a_{\mathbf{k}}^{(p)})(f f_{\mathbf{k}}) \eta_i^{-1}, \text{Ad}(w_i)^{-1}(c_{\mathbf{k}}^{(p)})u') \\ &= \sum_{(a', a'') \in \mathcal{A}_{\mathbf{k}}^{(p)}} \delta_i(b_{\mathbf{k}}^{(p)}, R'_i(a')(f f_{\mathbf{k}}) R'_i(a'')(\eta_i^{-1}), \text{Ad}(w_i)^{-1}c_{\mathbf{k}}^{(p)}u') \\ &= \sum_{(a', a'') \in \mathcal{A}_{\mathbf{k}}^{(p)}} -\eta(a'') \delta_i(b_{\mathbf{k}}^{(p)}, R'_i(a')(f f_{\mathbf{k}}) \eta_i^{-1}, \text{Ad}(w_i)^{-1}c_{\mathbf{k}}^{(p)}u') \end{aligned}$$

By the Poincaré-Birkhoff-Witt theorem, we have a direct decomposition $U(\text{Ad}(w_i)\mathfrak{p}) = U(\text{Ad}(w_i)\mathfrak{p})(\text{Ad}(w_i)\mathfrak{n}) \oplus U(\text{Ad}(w_i)\mathfrak{l})$. Hence we may assume that $c_{\mathbf{k}}^{(p)} \in U(\text{Ad}(w_i)\mathfrak{p})(\text{Ad}(w_i)\mathfrak{n})$ or $c_{\mathbf{k}}^{(p)} \in U(\text{Ad}(w_i)\mathfrak{l})$. If $c_{\mathbf{k}}^{(p)} \in U(\text{Ad}(w_i)\mathfrak{p})(\text{Ad}(w_i)\mathfrak{n})$ then this sum is equal to 0. If $c_{\mathbf{k}}^{(p)} \in U(\text{Ad}(w_i)\mathfrak{l})$ then $w_i^{-1} \text{wt } c_{\mathbf{k}}^{(p)}|_{\mathfrak{a}} = 0$. Hence,

$$\begin{aligned} & w_i^{-1}(\text{wt } b_{\mathbf{k}}^{(p)} + \text{wt}(R'_i(a') f f_{\mathbf{k}}))|_{\mathfrak{a}} \\ &= w_i^{-1}(\text{wt } c_{\mathbf{k}}^{(p)} + \text{wt } b_{\mathbf{k}}^{(p)} + \text{wt } a' + \text{wt } f + \text{wt } f_{\mathbf{k}})|_{\mathfrak{a}} \\ &= w_i^{-1}(\text{wt}((\text{ad}(e)^{\mathbf{k}}X)T) + \text{wt } f + \text{wt } f_{\mathbf{k}} - \text{wt } a'')|_{\mathfrak{a}} \\ &= w_i^{-1}(\text{wt } X + \text{wt } T + \text{wt } f - \text{wt } a'')|_{\mathfrak{a}} \\ &= \nu + w_i^{-1}(\text{wt } X - \text{wt } a'')|_{\mathfrak{a}} > \nu + w_i^{-1} \text{wt } X|_{\mathfrak{a}}. \end{aligned}$$

So we have

$$\begin{aligned} & \delta_i(b_{\mathbf{k}}^{(p)}, R'_i(-a_{\mathbf{k}}^{(p)})(f f_{\mathbf{k}} \eta_i^{-1}) - R'_i(-a_{\mathbf{k}}^{(p)})(f f_{\mathbf{k}}) \eta_i^{-1}, \text{Ad}(w_i)^{-1}(c_{\mathbf{k}}^{(p)})u') \\ & \in \sum_{\nu' > \nu} V(\nu' + w_i^{-1} \text{wt } X|_{\mathfrak{a}}). \end{aligned}$$

By the definition of $\widetilde{\eta}_i$, we have $\widetilde{R}_i(X')\widetilde{\eta}_i = 0$ for $X' \in \text{Ad}(w_i)\overline{\mathfrak{n}} \cap \overline{\mathfrak{n}}_0$. Hence by Lemma 8.2, we have

$$\begin{aligned} \delta_i(b_{\mathbf{k}}^{(p)}, R'_i(-a_{\mathbf{k}}^{(p)})(f f_{\mathbf{k}})\eta_i^{-1}, \text{Ad}(w_i)^{-1}(c_{\mathbf{k}}^{(p)})u') \\ = \delta_i(b_{\mathbf{k}}^{(p)}, R'_i(-a_{\mathbf{k}}^{(p)})(f f_{\mathbf{k}}), \text{Ad}(w_i)^{-1}(c_{\mathbf{k}}^{(p)})u')\widetilde{\eta}_i^{-1} \end{aligned}$$

Hence, we have

$$\begin{aligned} \sum_{\mathbf{k}, p} \delta_i(b_{\mathbf{k}}^{(p)}, R'_i(-a_{\mathbf{k}}^{(p)})(f f_{\mathbf{k}})\eta_i^{-1}, \text{Ad}(w_i)^{-1}(c_{\mathbf{k}}^{(p)})u') \\ = \sum_{\mathbf{k}, p} \delta_i(b_{\mathbf{k}}^{(p)}, R'_i(-a_{\mathbf{k}}^{(p)})(f f_{\mathbf{k}}), \text{Ad}(w_i)^{-1}(c_{\mathbf{k}}^{(p)})u')\widetilde{\eta}_i^{-1} \\ = \sum_{\mathbf{k}, p} \delta_i(a_{\mathbf{k}}^{(p)}b_{\mathbf{k}}^{(p)}c_{\mathbf{k}}^{(p)}, (f f_{\mathbf{k}}), u')\widetilde{\eta}_i^{-1} \\ = (X\delta_i(T, f, u'))\widetilde{\eta}_i^{-1}. \end{aligned}$$

We get (1).

(2) Take $T = 1$ in (1). Then we have

$$\delta_i(T, f\eta_i^{-1}, u') - (\delta_i(T, f\eta_i^{-1}, u'))\widetilde{\eta}_i^{-1} \in \sum_{\nu' > \nu} V(\nu').$$

Hence if

$$\sum_s \delta_i(S_s, h_s, v'_s)\widetilde{\eta}_i^{-1} \in \sum_{\nu' > \nu} V(\nu')$$

then

$$\delta_i(T, f\eta_i^{-1}, u') \in \sum_{\nu' > \nu} V(\nu')$$

However, by Lemma 3.2 (3), we have $V(\nu) \cap \sum_{\nu' \neq \nu} V(\nu') = 0$. This is a contradiction. \square

Proposition 8.4. *Let $\widetilde{\mu} \in (\mathfrak{h} \cap \mathfrak{m})^*$ be an infinitesimal character of σ . Assume that $I_i/I_{i-1} \neq 0$ and for all $\widetilde{w} \in \widetilde{W}$,*

$$\lambda - \widetilde{w}(\lambda + \widetilde{\mu})|_{\mathfrak{a}} \notin \mathbb{Z}_{\leq 0}((\Sigma^+ \setminus \Sigma_M^+) \cap w_i^{-1}\Sigma^+)|_{\mathfrak{a}} \setminus \{0\}.$$

Then

$$\text{Wh}_{\eta}(I'_i) = \{(\eta_i^{-1} \otimes u')\delta_i \mid u' \in \text{Wh}_{w_i^{-1}\eta}((\sigma \otimes e^{\lambda+\rho})')\}.$$

PROOF. Let $x = \sum_s \delta_i(T_s, f_s \eta_i^{-1}, u'_s)$ be an element of $\text{Wh}_\eta(I'_i)$ where $T_s \in U(\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \bar{\mathfrak{n}}_0)$, $f_s \in \mathcal{P}(O_i)$ and $u'_s \in J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$. For $X \in \text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0$, we have $(X - \eta(X))x = \sum_s \delta_i(T_s, (L_X - \eta(X))(f_s \eta_i^{-1}), u'_s) = \sum_s \delta_i(T_s, L_X(f_s) \eta_i^{-1}, u'_s)$ by Lemma 3.5. Hence, we may assume $f_s = 1$.

Let $z \in Z(\mathfrak{g})$. Since $J'_\eta(I(\sigma, \lambda))$ has an infinitesimal character $-(\lambda + \tilde{\mu})$, I'_i has the same character. Let $\chi(z)$ be a complex number such that z acts by $\chi(z)$ on I'_i . Take T_s and u'_s such that T_s are \mathfrak{a}_0 -weight vectors and lineally independent. Let $\nu = \min\{w_i^{-1} \text{wt } T_s|_{\mathfrak{a}}\}_s$. Then by Lemma 8.3 (1), we have

$$\begin{aligned} \chi(z)x &= zx = \gamma_2 \gamma_1(z)x \\ &\in \left(\gamma_3 \gamma_2 \gamma_1(z) \sum_{w_i^{-1} \text{wt } T_s|_{\mathfrak{a}} = \nu} \delta_i(T_s, 1, u'_s) \right) \tilde{\eta}_i^{-1} + \sum_{\nu' > \nu} V(\nu'). \end{aligned}$$

By Lemma 8.3 (1) ($T = 1$), we have

$$x \in \sum_{w_i^{-1} \text{wt } T_s|_{\mathfrak{a}} = \nu} \delta_i(T_s, 1, u'_s) \tilde{\eta}_i^{-1} + \sum_{\nu' > \nu} V(\nu').$$

Hence we have

$$\left((\chi(z) - \gamma_3 \gamma_2 \gamma_1(z)) \left(\sum_{w_i^{-1} \text{wt } T_s|_{\mathfrak{a}} = \nu} \delta_i(T_s, 1, u'_s) \right) \right) \tilde{\eta}_i^{-1} \in \sum_{\nu' > \nu} V(\nu').$$

By Lemma 8.3 (2), we have $(\chi(z) - \gamma_3 \gamma_2 \gamma_1(z)) \delta_i(T_s, 1, u'_s) = 0$ for all s such that $w_i^{-1} \text{wt } T_s|_{\mathfrak{a}} = \nu$. By the same calculation as that of the proof of Lemma 2.4, $H \delta_i(T_s, 1_u, u'_s) = (-w_i \lambda + \text{wt } T_s + \rho_0)(H) \delta_i(T_s, 1_u, u'_s)$ for $H \in \text{Ad}(w_i)\mathfrak{a}$. By Lemma 8.1, there exists a $\tilde{w} \in \tilde{W}$ such that $-\tilde{w}(\lambda + \tilde{\mu})|_{\text{Ad}(w_i)\mathfrak{a}} = -w_i \lambda + \text{wt } T_s$. Then $\lambda - w_i^{-1} \tilde{w}(\lambda + \tilde{\mu})|_{\mathfrak{a}} = w_i^{-1} \text{wt } T_s|_{\mathfrak{a}} \in \mathbb{Z}_{\leq 0}((\Sigma^+ \setminus \Sigma_M^+) \cap w_i^{-1} \Sigma^+)|_{\mathfrak{a}}$. By the assumption, $\text{wt } T_s = 0$, i.e., $T_s \in \mathbb{C}$. Hence, we may assume that x has a form $x = \delta_i(1, \eta_i^{-1}, u') + \sum_{s \geq 2} \delta_i(T_s, \eta_i^{-1}, u'_s)$ where $\text{wt } T_s \neq 0$ for all $s \geq 2$.

Take $X \in \mathfrak{n}_0 \cap \text{Ad}(w_i)\mathfrak{m}$. Then by Lemma 3.5 and the above claim,

$$0 = (X - \eta(X))x \in \delta_i(1, \eta_i^{-1}, (\text{Ad}(w_i)^{-1} X - \eta(X))u') + \sum_{\nu' > 0} V(\nu').$$

By Lemma 8.3, we have $\delta_i(1, \eta_i^{-1}, (\text{Ad}(w_i)^{-1} X - \eta(X))u') = 0$. Hence we have $u' \in \text{Wh}_{w_i^{-1}\eta}((\sigma \otimes e^{\lambda+\rho})')$. This implies that $x - \delta_i(1, \eta_i^{-1}, u') \in \text{Wh}_\eta(I'_i)$. If $x - \delta_i(1, \eta_i^{-1}, u') \neq 0$, then by the above argument, we have $\min\{w_i^{-1} \text{wt } T_s|_{\mathfrak{a}}\}_{s \geq 2} = 0$. This is a contradiction. \square

Theorem 8.5. *Assume that for all $w \in W(M)$ such that $\eta|_{wNw^{-1} \cap N_0} = 1$ the following two conditions hold:*

- (a) *For each exponent ν of σ and $\alpha \in \Sigma^+ \setminus w^{-1}(\Sigma^+ \cup \Sigma_\eta^-)$, we have $2\langle \alpha, \lambda + \nu \rangle / |\alpha|^2 \notin \mathbb{Z}_{\leq 0}$.*
- (b) *For all $\tilde{w} \in \tilde{W}$ we have $\lambda - \tilde{w}(\lambda + \tilde{\mu})|_{\mathfrak{a}} \notin \mathbb{Z}_{\leq 0}((\Sigma^+ \setminus \Sigma_M^+) \cap w^{-1}\Sigma^+)|_{\mathfrak{a}} \setminus \{0\}$ where $\tilde{\mu}$ is an infinitesimal character of σ .*

Moreover, assume that η is unitary. Then we have

$$\dim \text{Wh}_\eta(I(\sigma, \lambda)') = \sum_{w \in W(M), w(\Sigma^+ \setminus \Sigma_M^+) \cap \text{supp } \eta = \emptyset} \dim \text{Wh}_{w^{-1}\eta}((\sigma \otimes e^{\lambda+\rho})').$$

PROOF. By the exact sequence $0 \rightarrow I_{i-1} \rightarrow I_i \rightarrow I_i/I_{i-1} \rightarrow 0$, we have $0 \rightarrow \text{Wh}_\eta(I_{i-1}) \rightarrow \text{Wh}_\eta(I_i) \rightarrow \text{Wh}_\eta(I_i/I_{i-1})$. By Lemma 8.4, it is sufficient to prove that the last map $\text{Wh}_\eta(I_i) \rightarrow \text{Wh}_\eta(I_i/I_{i-1})$ is surjective.

Take $x \in \text{Wh}_\eta(I_i') \simeq \text{Wh}_\eta(I_i/I_{i-1})$. Then x is $(\eta_i \otimes u')\delta_i$ for some $u' \in \text{Wh}_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$. By Lemma 4.5, there exists a distribution $x_t \in I_i(\lambda + t\rho)$ with meromorphic parameter t such that $x_t|_{U_i}$ is holomorphic and $(x_t|_{U_i})|_{t=0} = x$. Moreover, $(X - \eta(X))x_t = 0$ for $X \in \mathfrak{n}_0$. By Proposition 4.4 and the condition (a), the distribution x_t is holomorphic at $t = 0$. Hence $x_0|_{U_i} = x$. The map $\text{Wh}_\eta(I_i) \rightarrow \text{Wh}_\eta(I_i/I_{i-1})$ is surjective. \square

Next we consider the module $\text{Wh}_\eta((I(\sigma, \lambda)_{K\text{-finite}})^*)$. Take a filtration $\tilde{I}_i \subset J_\eta^*(I(\sigma, \lambda))$ as in Theorem 7.3.

Lemma 8.6. *Let V be an object of the category \mathcal{O}' . Then we have $C(H^0(\mathfrak{n}_\eta, V)) = H^0(\mathfrak{n}_\eta, C(V))$ where $H^0(\mathfrak{n}_\eta, V) = \{v \in V \mid \mathfrak{n}_\eta v = 0\}$ is the 0-th \mathfrak{n}_η -cohomology.*

PROOF. We get the lemma by the following equation.

$$\begin{aligned} H^0(\mathfrak{n}_\eta, C(V)) &= H^0(\mathfrak{n}_\eta, D'(V)^*) = (D'(V)/\mathfrak{n}_\eta D'(V))^* \\ &= CD'(D'(V)/\mathfrak{n}_\eta D'(V)) = C(H^0(\mathfrak{n}_\eta, D'(V)^*)_{\mathfrak{h}\text{-finite}}) \\ &= C(H^0(\mathfrak{n}_\eta, D'D'(V))) = C(H^0(\mathfrak{n}_\eta, V)). \end{aligned}$$

\square

Lemma 8.7. *Let e_1, \dots, e_l be a basis of $\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0$ such that each e_s is a root vector and $\bigoplus_{s \leq t-1} \mathbb{C}e_s$ is an ideal of $\bigoplus_{s \leq t} \mathbb{C}e_s$. In $S_{w_i, 0}$ where 0 is the trivial representation, we have the following formulae.*

(1) For all $t = 1, \dots, l$,

$$\begin{aligned} e_t(e_1^{-1} \dots e_{t-1}^{-1} e_t^{-(k_t+1)} \dots e_l^{-(k_l+1)}) \\ = e_1^{-1} \dots e_{t-1}^{-1} e_t^{-k_t} e_{t+1}^{-(k_{t+1}+1)} \dots e_l^{-(k_l+1)} \end{aligned}$$

(2) Fix $t \in \{1, \dots, l\}$ such that $e_t \in \mathfrak{n}_\eta$. Assume that $k_s = 0$ for all $s < t$ such that $e_s \in \mathfrak{n}_\eta$. Then

$$\begin{aligned} e_t(e_1^{-(k_1+1)} \dots e_l^{-(k_l+1)}) \\ = e_1^{-(k_1+1)} \dots e_{t-1}^{-(k_{t-1}+1)} e_t^{-k_t} e_{t+1}^{-(k_{t+1}+1)} \dots e_l^{-(k_l+1)}. \end{aligned}$$

(3) $X(e_1^{-1} \dots e_l^{-1}) = (e_1^{-1} \dots e_l^{-1})X$ for $X \in \text{Ad}(w_i)\mathfrak{m} \cap \mathfrak{n}_0$.

PROOF. Let α_s be a restricted root corresponding to e_s .

(1) It is sufficient to prove $e_t(e_1^{-1} \dots e_{t-1}^{-1}) = (e_1^{-1} \dots e_{t-1}^{-1})e_t$ in $S_{e_1} \otimes_{U(\mathfrak{g})} \dots \otimes_{U(\mathfrak{g})} S_{e_{t-1}}$. Since $\bigoplus_{s=1}^{t-1} \mathbb{C}e_s$ is an ideal of $\bigoplus_{s=1}^t \mathbb{C}e_s$, we have

$$e_t(e_1^{-1} \dots e_{t-1}^{-1}) - (e_1^{-1} \dots e_{t-1}^{-1})e_t \in \bigoplus_{k_s \geq 0} \mathbb{C}e_1^{-(k_1+1)} \dots e_{t-1}^{-(k_{t-1}+1)}.$$

An \mathfrak{a}_0 -weight of the left hand side is $-\alpha_1 - \dots - \alpha_{t-1} + \alpha_t$. However, the set of \mathfrak{a}_0 -weights of the right hand side is $\{-(k_1+1)\alpha_1 - \dots - (k_{t-1}+1)\alpha_{t-1} \mid k_s \in \mathbb{Z}_{\geq 0}\}$. Hence each \mathfrak{a}_0 -weight appearing in the right hand side is less than that of the left hand side. This implies $e_t(e_1^{-1} \dots e_{t-1}^{-1}) - (e_1^{-1} \dots e_{t-1}^{-1})e_t = 0$.

(2) We prove $e_t(e_1^{-(k_1+1)} \dots e_{t-1}^{-(k_{t-1}+1)}) = (e_1^{-(k_1+1)} \dots e_{t-1}^{-(k_{t-1}+1)})e_t$ in $S_{e_1} \otimes_{U(\mathfrak{g})} \dots \otimes_{U(\mathfrak{g})} S_{e_{t-1}}$. As in the proof of (1), we have

$$\begin{aligned} e_t(e_1^{-(k_1+1)} \dots e_{t-1}^{-(k_{t-1}+1)}) - (e_1^{-(k_1+1)} \dots e_{t-1}^{-(k_{t-1}+1)})e_t \\ \in \bigoplus_{k_s \geq 0} \mathbb{C}e_1^{-(k_1+1)} \dots e_{t-1}^{-(k_{t-1}+1)}. \end{aligned}$$

An \mathfrak{a}_η -weight of the left hand side is $\sum_{e_s \in \mathfrak{n}_\eta, s < t} -\alpha_s + \alpha_t$. However, the set of \mathfrak{a}_η -weights of the right hand side is $\{\sum_{e_s \in \mathfrak{n}_\eta, s < t} -(k_s+1)\alpha_s \mid k_s \in \mathbb{Z}_{\geq 0}\}$. Hence each \mathfrak{a}_η -weight appearing in the right hand side is less than that of the left hand side. This implies the lemma.

(3) We may assume X is a restricted root vector. Let α be a restricted root of X . Since X normalizes $\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0$, we have

$$X(e_1^{-1} \dots e_l^{-1}) - (e_1^{-1} \dots e_l^{-1})X \in \bigoplus_{k_s \geq 0} \mathbb{C}e_1^{-(k_1+1)} \dots e_l^{-(k_l+1)}.$$

Then $X(e_1^{-1} \cdots e_l^{-1}) - (e_1^{-1} \cdots e_l^{-1})X$ has an \mathfrak{a}_0 -weight $-(\alpha_1 + \cdots + \alpha_s) + \alpha$. However, $e_1^{-(k_1+1)} \cdots e_l^{-(k_l+1)}$ has a \mathfrak{a}_0 -weight $-((k_1+1)\alpha_1 + \cdots + (k_l+1)\alpha_l) < -(\alpha_1 + \cdots + \alpha_s) + \alpha$. Hence $X(e_1^{-1} \cdots e_l^{-1}) - (e_1^{-1} \cdots e_l^{-1})X = 0$. \square

Lemma 8.8. *Let e_1, \dots, e_l be a basis of $\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0$ such that e_s is a root vector and $\bigoplus_{s \leq t-1} \mathbb{C}e_s$ is an ideal of $\bigoplus_{s \leq t} \mathbb{C}e_s$. Let V be a $U(\mathfrak{m} \oplus \mathfrak{a})$ -representation. Regard V as a \mathfrak{p} -representation by $\mathfrak{n}V = 0$. By Lemma 6.3, we have $T_{w_i}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V) \simeq (\bigoplus_{k_s \geq 0} \mathbb{C}e_1^{-(k_1+1)} \cdots e_l^{-(k_l+1)}) \otimes U(\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \bar{\mathfrak{n}}_0) \otimes w_i V$. Then we have $\{v \in e_1^{-1} \cdots e_l^{-1} \otimes 1 \otimes w_i V \mid \mathfrak{n}_\eta v = 0\} = e_1^{-1} \cdots e_l^{-1} \otimes 1 \otimes H^0(\text{Ad}(w_i)\mathfrak{m} \cap \mathfrak{n}_\eta, w_i V)$.*

PROOF. Take $v = e_1^{-1} \cdots e_l^{-1} \otimes 1 \otimes v_0 \in H^0(\mathfrak{n}_\eta, T_{w_i}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V))$. Then for $X \in \text{Ad}(w_i)\mathfrak{m} \cap \mathfrak{n}_\eta$ we have $X(e_1^{-1} \cdots e_l^{-1} \otimes 1 \otimes v_0) = 0$. By Lemma 8.7, we have $e_1^{-1} \cdots e_l^{-1} \otimes 1 \otimes Xv_0 = 0$. Hence $Xv_0 = 0$. \square

By the definition of the Harish-Chandra homomorphism, we get the following lemma.

Lemma 8.9. *Let \mathfrak{q} be a parabolic subalgebra of \mathfrak{g} containing $\mathfrak{h} \oplus \mathfrak{u}_0$. Take a Levi decomposition $\mathfrak{l} \oplus \mathfrak{u}_\mathfrak{q}$ of \mathfrak{q} such that $\mathfrak{h} \subset \mathfrak{l}$. Let $\widetilde{W}_\mathfrak{l} \subset \widetilde{W}$ be the Weyl group of \mathfrak{l} , V an \mathfrak{l} -module with an infinitesimal character $\tilde{\mu}$. Put $V' = H^0(\mathfrak{u}_\mathfrak{q}, V)$ and $\widetilde{\rho}_{\mathfrak{u}_\mathfrak{q}}(H) = (1/2) \text{Tr ad}(H)|_{\mathfrak{u}_\mathfrak{q}}$ for $H \in \mathfrak{h}$. Then V' is \mathfrak{l} -stable and $V' = \bigoplus_{\tilde{w} \in \widetilde{W}_\mathfrak{l} \setminus \widetilde{W}} (V')_{[\tilde{w}\tilde{\mu} - \widetilde{\rho}_{\mathfrak{u}_\mathfrak{q}}]}$ where $(V')_{[\tilde{w}\tilde{\mu} - \widetilde{\rho}_{\mathfrak{u}_\mathfrak{q}}]}$ is the maximal \mathfrak{l} -submodule which has an infinitesimal character $\tilde{w}\tilde{\mu} - \widetilde{\rho}_{\mathfrak{u}_\mathfrak{q}}$. In particular, for an \mathfrak{l} -submodule V'' of V' , a highest weight of V'/V'' belongs to $\{\tilde{w}\tilde{\mu} - \tilde{\rho} \mid \tilde{w} \in \widetilde{W}\}$.*

The following lemma is well-known.

Lemma 8.10. *Let $V \in \mathcal{O}'$. Assume that V has an infinitesimal character $\tilde{\lambda} \in \mathfrak{h}^*$. Then a \mathfrak{h} -weight appearing in V is contained in $\{\tilde{w}\tilde{\lambda} - \tilde{\rho} - \alpha \mid \tilde{w} \in \widetilde{W}, \alpha \in \mathbb{Z}_{\geq 0}\Delta^+\}$.*

Now we determine the dimension of the space of Whittaker vectors of $\widetilde{I}_i/\widetilde{I}_{i-1}$ under some conditions.

Lemma 8.11. *Let $\tilde{\mu}$ be an infinitesimal character of σ . Assume that for all $\tilde{w} \in \widetilde{W} \setminus \widetilde{W}_M$, $(\lambda + \tilde{\mu}) - \tilde{w}(\lambda + \tilde{\mu}) \notin \mathbb{Z}\Delta$. Then we have $\dim \text{Wh}_\eta(\widetilde{I}_i/\widetilde{I}_{i-1}) = \dim \text{Wh}_{w_i^{-1}\eta}((\sigma_M \cap K\text{-finite})^*)$.*

PROOF. Put $V = T_{w_i}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} J^*(\sigma \otimes e^{\lambda+\rho}))$. By Theorem 7.3, we have $\text{Wh}_\eta(\widetilde{I}_i/\widetilde{I}_{i-1}) = \text{Wh}_\eta(C(V))$. Let e_1, \dots, e_l be a basis of $\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0$

such that $\bigoplus_{s \leq t-1} \mathbb{C}e_s$ is an ideal of $\bigoplus_{s \leq t} \mathbb{C}e_s$. Moreover, assume that each e_i is a root vector. For $\mathbf{k} = (k_1, \dots, k_l) \in \mathbb{Z}^l$, put $e^{\mathbf{k}} = e_1^{k_1} \cdots e_l^{k_l}$. Set $\mathbf{1} = (1, \dots, 1) \in \mathbb{Z}^l$. Then we have

$$V = \bigoplus_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^l} \mathbb{C}e^{-(\mathbf{k}+\mathbf{1})} \otimes U(\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \bar{\mathfrak{n}}_0) \otimes w_i J^*(\sigma \otimes e^{\lambda+\rho}).$$

Put

$$V' = \bigoplus_{\mathbf{k} \in \mathcal{A}} e^{-(\mathbf{k}+\mathbf{1})} \otimes U(\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \bar{\mathfrak{n}}_0 \cap \mathfrak{m}_\eta) \otimes H^0(\mathfrak{m} \cap \mathfrak{n}_\eta, w_i J^*(\sigma \otimes e^{\lambda+\rho}))$$

where $\mathcal{A} = \{(k_1, \dots, k_l) \in \mathbb{Z}_{\geq 0}^l \mid \text{if } e_i \in \mathfrak{n}_\eta \text{ then } k_i = 0\}$. It is easy to see that V' is an $\mathfrak{m}_\eta \oplus \mathfrak{a}_\eta$ -stable and $V' \subset H^0(\mathfrak{n}_\eta, V)$. We prove that $V' = H^0(\mathfrak{n}_\eta, V)$.

To prove $V' = H^0(\mathfrak{n}_\eta, V)$, it is sufficient to prove that there exists no highest weight vector in $H^0(\mathfrak{n}_\eta, V)/V'$. Let $v \in H^0(\mathfrak{n}_\eta, V)$ such that $(\mathfrak{m}_\eta \cap \mathfrak{u})v \in V'$.

First, we prove that $v \in e^{-\mathbf{1}} \otimes U(\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \bar{\mathfrak{n}}_0) \otimes J^*(\sigma \otimes e^{\lambda+\rho}) + V'$. Take $y_{\mathbf{k}} \in U(\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \bar{\mathfrak{n}}_0) \otimes J^*(\sigma \otimes e^{\lambda+\rho})$ such that $v = \sum_{\mathbf{k}} e^{-(\mathbf{k}+\mathbf{1})} \otimes y_{\mathbf{k}}$. We prove that if $k_t \neq 0$ and $e_t \in \mathfrak{n}_\eta$ then $y_{\mathbf{k}} = 0$ by induction on t where $\mathbf{k} = (k_1, \dots, k_l)$. Put $\mathbf{1}_t = (\delta_{st})_{1 \leq s \leq l} \in \mathbb{Z}^l$ (δ_{st} is Kronecker's delta). By inductive hypothesis, for $s < t$ such that $e_s \in \mathfrak{n}_\eta$, if $y_{\mathbf{k}} \neq 0$ then $k_s = 0$. By Lemma 8.7 (2), we have $e_t v = \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^l} e^{-(\mathbf{k}+\mathbf{1})+\mathbf{1}_t} \otimes y_{\mathbf{k}}$. Since $v \in H^0(\mathfrak{n}_\eta, V)$, we have $e_t v = 0$. Hence if $e^{-(\mathbf{k}+\mathbf{1})+\mathbf{1}_t} \neq 0$ then $y_{\mathbf{k}} = 0$. Since $e^{-(\mathbf{k}+\mathbf{1})+\mathbf{1}_t} = 0$ if and only if $k_t = 0$, $k_t \neq 0$ implies $y_{\mathbf{k}} = 0$. We prove that if $k_t \neq 0$ then $e^{-(\mathbf{k}+\mathbf{1})} \otimes y_{\mathbf{k}} \in V'$ by induction on t . If $e_t \in \mathfrak{n}_\eta$ then this claim is already proved. We may assume that $e_t \in \mathfrak{m}_\eta$. Hence $e_t V' \subset V'$. By inductive hypothesis, if $k_s \neq 0$ for some $s < t$ then $e^{-(\mathbf{k}+\mathbf{1})} \otimes y_{\mathbf{k}} \in V'$. Then we have $e_t v \in \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^l} e^{-(\mathbf{k}+\mathbf{1})+\mathbf{1}_t} \otimes y_{\mathbf{k}} + V'$ by Lemma 8.7 (1). Since $e_t v \in V'$, we have $\sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^l} e^{-(\mathbf{k}+\mathbf{1})+\mathbf{1}_t} \otimes y_{\mathbf{k}} \in V'$. By the definition of V' , if $e^{-(\mathbf{k}+\mathbf{1})+\mathbf{1}_t} \neq 0$ then $e^{-(\mathbf{k}+\mathbf{1})} \otimes y_{\mathbf{k}} \in V'$. Hence we get the claim.

We may assume that v is a weight vector with respect to \mathfrak{h} . We can take $\tilde{w} \in \tilde{W}$ such that $-\tilde{w}(\lambda + \tilde{\mu}) - \tilde{\rho}$ is a \mathfrak{h} -weight of v by Lemma 8.9. Put $\tilde{\rho}_M = \sum_{\alpha \in \Delta_M^+} (1/2)\alpha$. Since $J^*(\sigma \otimes e^{\lambda+\rho})$ has an infinitesimal character $-(\lambda + \tilde{\mu} + \rho)$, a \mathfrak{h} -weight appearing in $J^*(\sigma \otimes e^{\lambda+\rho})$ is contained in $\{-\tilde{w}(\lambda + \tilde{\mu} + \rho) - \tilde{\rho}_M + \alpha \mid \tilde{w} \in \tilde{W}_M, \alpha \in \mathbb{Z}\Delta_M\}$ by Lemma 8.10. Since $-\rho \in \mathfrak{a}^*$, we have $\tilde{w}\rho = \rho$ for $\tilde{w} \in \tilde{W}_M$. Hence we have $-\tilde{w}\rho - \tilde{\rho}_M = -\rho - \tilde{\rho}_M = -\tilde{\rho}$. Notice that $w_i \tilde{\rho} - \tilde{\rho} \in \mathbb{Z}\Delta$. Therefore a \mathfrak{h} -weight appearing in V is contained

in

$$\begin{aligned} & -w_i \widetilde{W}_M(\lambda + \tilde{\mu}) - w_i \tilde{\rho} + w_i \mathbb{Z} \Delta_M + \mathbb{Z}_{\geq 0}(w_i \Delta^- \cap \Delta^-) - \mathbb{Z}_{\geq 1}(w_i \Delta^- \cap \Delta^+) \\ & \subset -w_i \widetilde{W}_M(\lambda + \tilde{\mu}) - \tilde{\rho} + \mathbb{Z} \Delta. \end{aligned}$$

This implies that for some $\tilde{w}' \in \widetilde{W}_M$, we have $\tilde{w}(\lambda + \tilde{\mu}) - w_i \tilde{w}'(\lambda + \tilde{\mu}) \in \mathbb{Z} \Delta$. By the assumption we have $\tilde{w} \in w_i \widetilde{W}_M$. This implies $(\text{wt } v)(\text{Ad}(w_i)H) = -(\lambda(H) + w_i^{-1} \tilde{\rho}(H))$ for all $H \in \mathfrak{a}$ where $\text{wt } v$ is a \mathfrak{h} -weight of v .

Take $T_p \in U(\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \bar{\mathfrak{n}}_0)$ and $x_p \in w_i J^*(\sigma \otimes e^{\lambda+\rho})$ such that $v \in \sum_p e^{-1} \otimes T_p \otimes x_p + V'$. We may assume that T_p (resp. x_p) is a \mathfrak{h} -weight vector with respect to the adjoint action (resp. the action induced from $\sigma \otimes e^{\lambda+\rho}$). We denote its \mathfrak{h} -weight by $\text{wt } T_p$ and $\text{wt } x_p$. Fix $H \in \mathfrak{a}$. Then $\alpha(H) = 0$ for all $\alpha \in \Delta_M$. Since $\text{wt } x_p \in -w_i(\widetilde{W}_M(\lambda + \tilde{\mu}) + \tilde{\rho} + \mathbb{Z} \Delta_M)$, $(\text{wt } x_p)(\text{Ad}(w_i)H) = -(\lambda + \tilde{\rho})(H)$. Hence

$$\begin{aligned} & (\text{wt } v)(\text{Ad}(w_i)H) \\ & = (\text{wt}(e^{-1}) + \text{wt}(T_p) + \text{wt}(x_p))(\text{Ad}(w_i)H) \\ & = (\text{wt}(e^{-1})(\text{Ad}(w_i)H) + (\text{wt } T_p)(\text{Ad}(w_i)H) - (\lambda + \tilde{\rho})(H)) \\ & = (\text{wt}(e^{-1})(\text{Ad}(w_i)H) + (\text{wt } T_p)(\text{Ad}(w_i)H) - (\lambda + \tilde{\rho})(H)). \end{aligned}$$

We calculate $\text{wt}(e^{-1})(\text{Ad}(w_i)H)$. By the definition, $\text{wt}(e^{-1})(\text{Ad}(w_i)H) = \text{Tr ad}(\text{Ad}(w_i)H)|_{\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0}$. Since we have $\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0 = \text{Ad}(w_i)\bar{\mathfrak{n}}_0 \cap \mathfrak{n}_0$, we have

$$\begin{aligned} \text{Tr ad}(\text{Ad}(w_i)H)|_{\text{Ad}(w_i)\bar{\mathfrak{n}} \cap \mathfrak{n}_0} & = \text{Tr ad}(\text{Ad}(w_i)H)|_{\text{Ad}(w_i)\bar{\mathfrak{n}}_0 \cap \mathfrak{n}_0} \\ & = \text{Tr ad}(H)|_{\text{Ad}(w_i)^{-1}\mathfrak{n}_0 \cap \bar{\mathfrak{n}}_0} = (-\tilde{\rho} + w_i^{-1} \tilde{\rho})(H). \end{aligned}$$

Hence we get

$$(\text{wt } v)(\text{Ad}(w_i)H) = (\text{wt } T_p)(\text{Ad}(w_i)H) - (\lambda + w_i^{-1} \tilde{\rho})(H).$$

We have already proved that $(\text{wt } v)(\text{Ad}(w_i)H) = -(\lambda + w_i^{-1} \tilde{\rho})(H)$. Therefore we get $(\text{wt } T_p)(\text{Ad}(w_i)H) = 0$ for all $H \in \mathfrak{a}$. Since $T_p \in U(\text{Ad}(w_i)\bar{\mathfrak{n}})$, this implies $T_p \in \mathbb{C}$, i.e., there exist $v' \in e_1^{-1} \cdots e_l^{-1} \otimes 1 \otimes w_i J^*(\sigma \otimes e^{\lambda+\rho})$ and $v'' \in V'$ such that $v = v' + v''$. Therefore $\mathfrak{n}_\eta(v') = \mathfrak{n}_\eta(v - v'') = 0$. Hence, $v' \in V'$ by Lemma 8.8. Therefore $H^0(\mathfrak{n}_\eta, V) = V'$.

For an $\mathfrak{m}_0 \oplus \mathfrak{a}_0$ -module τ and a subalgebra \mathfrak{c} of \mathfrak{g} containing $\mathfrak{m}_0 \oplus \mathfrak{a}_0$, put $M_{\mathfrak{c}}(\tau) = U(\mathfrak{c}) \otimes_{U(\mathfrak{c} \cap \bar{\mathfrak{p}}_0)} (\tau \otimes \rho')$ where $\bar{\mathfrak{n}}_0 \cap \mathfrak{c}$ acts on τ trivially and $\rho'(H) = (\text{Tr}(\text{ad}(H)|_{\mathfrak{c} \cap \bar{\mathfrak{n}}_0}))/2$ for $H \in \mathfrak{a}_0$.

For $\tilde{\lambda} \in \mathfrak{h}^*$ such that $\tilde{\lambda}|_{\mathfrak{m}_0}$ is regular dominant integral, let $\sigma_{M_0 A_0, \tilde{\lambda}}$ be the finite-dimensional representation of $M_0 A_0$ with an infinitesimal character $\tilde{\lambda}$. Let $\text{ch } M$ be the character of M . We can take integers $c_{\tilde{\lambda}}$ such that

$$\begin{aligned} \text{ch } D' H^0(\mathfrak{n}_\eta \cap \text{Ad}(w_i)\mathfrak{m}, w_i J^*(\sigma \otimes e^{\lambda+\rho})) \\ = \sum_{\tilde{\lambda}} c_{\tilde{\lambda}} \text{ch } M_{(\mathfrak{m}_\eta \cap \text{Ad}(w_i)\mathfrak{m}) + \mathfrak{a}_0}(\sigma_{M_0 A_0, \tilde{\lambda}}) \end{aligned}$$

Then we have $\text{ch } D' V' = \sum_{\tilde{\lambda}} c_{\tilde{\lambda}} \text{ch } M_{\mathfrak{m}_\eta \oplus \mathfrak{a}_\eta}(\sigma_{M_0 A_0, \tilde{\lambda}})$. The functor $X \mapsto \text{Wh}_{\eta|_{\mathfrak{m}_\eta \cap \mathfrak{n}_0}}(X^*)$ is an exact functor by a result of Lynch [Lyn79]. Hence, we have $\dim \text{Wh}_{\eta|_{\mathfrak{m}_\eta \cap \mathfrak{n}_0}}(C(V')) = \sum_{\tilde{\lambda}} c_{\tilde{\lambda}} \dim \text{Wh}_{\eta|_{\mathfrak{m}_\eta \cap \mathfrak{n}_0}}(M_{\mathfrak{m}_\eta \oplus \mathfrak{a}_\eta}(\sigma_{M_0 A_0, \tilde{\lambda}})^*)$. Lynch also proves $\dim \text{Wh}_{\eta|_{\mathfrak{m}_\eta \cap \mathfrak{n}_0}}(M_{\mathfrak{m}_\eta}(\sigma_{M_0 A_0, \tilde{\lambda}})^*) = \dim \sigma_{M_0 A_0, \tilde{\lambda}}$. Therefore, by Lemma 8.6, we have $\dim \text{Wh}_\eta(\tilde{I}_i / \tilde{I}_{i-1}) = \dim \text{Wh}_{\eta|_{\mathfrak{m}_\eta \cap \mathfrak{n}_0}}(C(V')) = \sum_{\tilde{\lambda}} c_{\tilde{\lambda}} \dim \sigma_{M_0 A_0, \tilde{\lambda}}$. By the same argument we have

$$\begin{aligned} & \sum_{\tilde{\lambda}} c_{\tilde{\lambda}} \dim \sigma_{M_0 A_0, \tilde{\lambda}} \\ &= \sum_{\tilde{\lambda}} c_{\tilde{\lambda}} \dim \text{Wh}_{\eta|_{\mathfrak{m}_\eta \cap \text{Ad}(w_i)\mathfrak{m} \cap \mathfrak{n}_0}}(M_{(\mathfrak{m}_\eta \cap \text{Ad}(w_i)\mathfrak{m}) + \mathfrak{a}_0}(\sigma_{M_0 A_0, \tilde{\lambda}})^*) \\ &= \dim \text{Wh}_{\eta|_{\mathfrak{m}_\eta \cap \text{Ad}(w_i)\mathfrak{m} \cap \mathfrak{n}_0}}(CH^0(\mathfrak{n}_\eta \cap \text{Ad}(w_i)\mathfrak{m}, w_i J^*(\sigma \otimes e^{\lambda+\rho}))) \\ &= \dim \text{Wh}_{\eta|_{\text{Ad}(w_i)\mathfrak{m} \cap \mathfrak{n}_0}}(C(w_i J^*(\sigma \otimes e^{\lambda+\rho}))) \\ &= \dim \text{Wh}_{w_i^{-1}\eta}(C(J^*(\sigma \otimes e^{\lambda+\rho}))) \\ &= \dim \text{Wh}_{w_i^{-1}\eta}((\sigma_M \cap K\text{-finite})^*). \end{aligned}$$

This implies the lemma. \square

Theorem 8.12. *Let $\tilde{\mu}$ be an infinitesimal character of σ . Assume that for all $\tilde{w} \in \widetilde{W} \setminus \widetilde{W}_M$, $(\lambda + \tilde{\mu}) - \tilde{w}(\lambda + \tilde{\mu}) \notin \mathbb{Z}\Delta$. Then we have*

$$\dim \text{Wh}_\eta((I(\sigma, \lambda)_{K\text{-finite}})^*) = \sum_{w \in W(M)} \dim \text{Wh}_{w^{-1}\eta}((\sigma_M \cap K\text{-finite})^*).$$

PROOF. Since a \mathfrak{h} -weight appearing in $T_{w_i}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} J^*(\sigma \otimes e^{\lambda+\rho}))$ belongs to $\{-w_i \tilde{w}(\lambda + \tilde{\mu}) - \tilde{\rho} + \alpha \mid \tilde{w} \in \widetilde{W}_M, \alpha \in \Delta\}$, the exact sequence $0 \rightarrow I_{i-1} \rightarrow I_i \rightarrow T_{w_i}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} J^*(\sigma \otimes e^{\lambda+\rho})) \rightarrow 0$ splits. Hence, we have $J_\eta^*(I(\sigma, \lambda)) = \bigoplus_i \Gamma_\eta(C(T_{w_i}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} J^*(\sigma \otimes e^{\lambda+\rho})))$. Therefore the theorem follows from Lemma 8.11. \square

Finally we study the case of σ is finite-dimensional. If σ is finite-dimensional, then $\mathfrak{m} \cap \mathfrak{n}_0$ acts on σ nilpotently. Hence $\text{Wh}_{w_i^{-1}\eta}(\sigma^*) \neq 0$ if and only if $w_i^{-1}\eta = 0$ on $\mathfrak{m} \cap \mathfrak{n}_0$.

Definition 8.13. Let $\Theta, \Theta_1, \Theta_2$ be subsets of Π .

- (1) Put $W(\Theta) = \{w \in W \mid w(\Theta) \subset \Sigma^+\}$ and $\Sigma_\Theta = \mathbb{Z}\Theta \cap \Sigma$.
- (2) Put $W(\Theta_1, \Theta_2) = \{w \in W(\Theta_1) \cap W(\Theta_2)^{-1} \mid w(\Sigma_{\Theta_1}) \cap \Sigma_{\Theta_2} = \emptyset\}$.
- (3) Let W_Θ be the Weyl group of Σ_Θ .

Lemma 8.14. Let Θ be a subset of Π corresponding to P .

- (1) We have $\#W(\text{supp } \eta, \Theta) = \#\{w \in W(M) \mid w(\Sigma^+) \cap \Sigma_\eta^+ = \emptyset\}$.
- (2) We have $\#W(\text{supp } \eta, \Theta) \times \#W_{\text{supp } \eta} = \#\{w \in W(M) \mid \text{supp } \eta \cap w(\Sigma_M^+) = \emptyset\}$.

PROOF. (1) Put $\mathcal{W} = \{w \in W(M) \mid w(\Sigma^+) \cap \Sigma_\eta^+ = \emptyset\}$. Let $w_{\eta,0}$ be the longest Weyl element of W_{M_η} . We prove that the map $\mathcal{W} \rightarrow W(\text{supp } \eta, \Theta)$ defined by $w \mapsto (w_{\eta,0}w)^{-1}$ is well-defined and bijective.

First we prove that the map is well-defined. Let $w \in \mathcal{W}$. The equation $w(\Sigma^+) \cap \Sigma_\eta^+ = \emptyset$ implies that $(w_{\eta,0}w)^{-1}(\Sigma_\eta^+) \subset \Sigma^+$. Hence, $(w_{\eta,0}w)^{-1} \in W(\text{supp } \eta)$. Moreover, $w(\Sigma_M^+) \subset \Sigma^+$ and $w(\Sigma^+) \cap \Sigma_\eta^+ = \emptyset$ imply that $w(\Sigma_M^+) \subset \Sigma^+ \cap (\Sigma \setminus \Sigma_\eta^+) = \Sigma^+ \setminus \Sigma_\eta^+$. Hence, $(w_{\eta,0}w)(\Sigma_M^+) \subset \Sigma^+ \setminus \Sigma_\eta^+ \subset \Sigma^+$. We have $(w_{\eta,0}w)^{-1} \in W(\Theta)^{-1}$. Finally $w(\Sigma_M^+) \subset \Sigma^+ \setminus \Sigma_\eta^+$ implies $w(\Sigma) \subset \Sigma \setminus \Sigma_\eta$. Hence we have $(w_{\eta,0}w)^{-1}\Sigma_\eta \cap \Sigma_M = w^{-1}\Sigma_\eta \cap \Sigma_M = \emptyset$.

Assume that $(w_{\eta,0}w)^{-1} \in W(\text{supp } \eta, \Theta)$. Then $(w_{\eta,0}w)^{-1}(\Sigma_\eta^+) \subset \Sigma^+$ implies that $w(\Sigma^+) \cap \Sigma_\eta^+ = \emptyset$. Since $(w_{\eta,0}w)^{-1}\Sigma_\eta \cap \Sigma_M = \emptyset$ we have $w(\Sigma_M) \cap \Sigma_\eta = \emptyset$. By $(w_{\eta,0}w)(\Sigma_M^+) \subset \Sigma^+$ and $w(\Sigma^+) \cap \Sigma_\eta^+ = \emptyset$, we have $w(\Sigma_M^+) \subset ((\Sigma^+ \setminus \Sigma_\eta^+) \cup \Sigma_\eta^-) \cap (\Sigma \setminus \Sigma_\eta^-) = (\Sigma^+ \setminus \Sigma_\eta^+)$. Consequently we have $w \in W(M)$.

(2) Put $\mathcal{W} = \{w \in W(M) \mid \text{supp } \eta \cap w(\Sigma_M^+) = \emptyset\}$. Define the map $\varphi: W(\text{supp } \eta, \Theta) \times W_{\text{supp } \eta} \rightarrow \mathcal{W}$ by $(w_1, w_2) \mapsto w_2w_1^{-1}$. This map is injective since $W(\text{supp } \eta, \Theta) \subset W(\text{supp } \eta)$. We prove that φ is well-defined and surjective. Since $w_1^{-1}(\Sigma_M^+) = w_1^{-1}(\Sigma_M^+) \cap \Sigma^+ \subset \Sigma^+ \setminus \Sigma_\eta^+$, we have $w_2w_1^{-1}(\Sigma_M^+) \subset \Sigma^+ \setminus \Sigma_\eta^+$. Hence, φ is well-defined. Next let $w \in \mathcal{W}$. Let $w_1 \in W(\text{supp } \eta)^{-1}$ and $w_2 \in W_{\text{supp } \eta}$ such that $w = w_2w_1^{-1}$. Then $w_1^{-1}(\Sigma_M^+) = w_2^{-1}w(\Sigma_M^+) \subset w_2^{-1}(\Sigma^+ \setminus \Sigma_\eta^+) = \Sigma^+ \setminus \Sigma_\eta^+$. This implies $w_1 \in W(\text{supp } \eta, \Theta)$. \square

Lemma 8.15. *Assume that σ is irreducible and finite-dimensional. Let $\tilde{\mu}$ be the highest weight of σ and V the irreducible finite-dimensional representation of M_0A_0 with highest weight $\lambda + \tilde{\mu}$. Then we have $\sigma/(\mathfrak{m} \cap \mathfrak{n}_0)\sigma \simeq V$ as an M_0A_0 -module. In particular, $\dim \text{Wh}_0(\sigma') = \dim V$.*

PROOF. We prove that $\text{Wh}_0(\sigma^*) \simeq V^*$. Let $\tilde{w}_{M,0}$ be the longest element of \widetilde{W}_M . Then both sides have a highest weight $-\tilde{w}_{M,0}(\tilde{\mu} + \lambda)$ and the space of highest weight vectors are 1-dimensional. \square

As an Corollary of Theorem 8.5 and Theorem 8.12, we have the following theorem announced by T. Oshima. Define $\widetilde{\rho}_M \in \mathfrak{h}^*$ by $\widetilde{\rho}_M = (1/2) \sum_{\alpha \in \Delta_M^+} \alpha$.

Theorem 8.16. *Assume that σ is the irreducible finite-dimensional representation with highest weight $\tilde{\nu}$. Let $\dim_M(\lambda + \tilde{\nu})$ be a dimension of the finite-dimensional irreducible representation of M_0A_0 with highest weight $\lambda + \tilde{\nu}$.*

(1) *Assume that for all $w \in W$ such that $\eta|_{wN_0w^{-1} \cap N_0} = 1$ the following two conditions hold:*

- (a) *For all $\alpha \in \Sigma^+ \setminus w^{-1}(\Sigma_M^+ \cup \Sigma_\eta^+)$ we have $2\langle \alpha, \lambda + w_0\tilde{\nu} \rangle / |\alpha|^2 \notin \mathbb{Z}_{\leq 0}$.*
- (b) *For all $\tilde{w} \in \widetilde{W}$ we have $\lambda - \tilde{w}(\lambda + \tilde{\nu} + \widetilde{\rho}_M)|_{\mathfrak{a}} \notin \mathbb{Z}_{\leq 0}((\Sigma^+ \setminus \Sigma_M^+) \cap w^{-1}\Sigma^+)|_{\mathfrak{a}} \setminus \{0\}$.*

Then we have

$$\dim \text{Wh}_\eta(I(\sigma, \lambda)') = \#W(\text{supp } \eta, \Theta) \times (\dim_M(\lambda + \tilde{\nu}))$$

(2) *Assume that for all $\tilde{w} \in \widetilde{W} \setminus \widetilde{W}_M$, $(\lambda + \tilde{\nu}) - \tilde{w}(\lambda + \tilde{\nu}) \notin \Delta$. Then we have*

$$\begin{aligned} \dim \text{Wh}_\eta((I(\sigma, \lambda)_{K\text{-finite}})^*) \\ = \#W(\text{supp } \eta, \Theta) \times \#W_{\text{supp } \eta} \times (\dim_M(\lambda + \tilde{\nu})) \end{aligned}$$

§A. C^∞ -function with values in Fréchet space

§A.1. \mathcal{L} -distributions and tempered \mathcal{L} -distributions

Let M be a C^∞ -manifold, V a Fréchet space and \mathcal{L} a vector bundle on M with fibers V . We define the sheaf of \mathcal{L} -distributions as follows.

First we assume that \mathcal{L} is trivial on M . Then the definition of \mathcal{L} -distributions is found in Kolk-Varadarajan [KV96] and it is easy to see that \mathcal{L} -distributions makes a sheaf on M .

In general, let $M = \bigcup_{\lambda \in \Lambda} U_\lambda$ be an open covering of M such that on each U_λ the vector bundle \mathcal{L} is trivial. For an arbitrary open subset U of M , put

$$\mathcal{D}'(U, \mathcal{L}) = \left\{ (x_\lambda) \in \prod_{\lambda \in \Lambda} \mathcal{D}'(U \cap U_\lambda, \mathcal{L}) \mid x_\lambda = x_{\lambda'} \text{ on } U_\lambda \cap U_{\lambda'} \right\}.$$

It is independent of the choice of an open covering $\{U_\lambda\}$ and defines the sheaf of \mathcal{L} -distributions on M .

Now assume that M has a compactification X , i.e., M is an open dense subset of a compact manifold X . In this case, we define a subspace $\mathcal{T}(M, \mathcal{L})$ of $\mathcal{D}'(M, \mathcal{L})$ by

$$\mathcal{T}(M, \mathcal{L}) = \{x \in \mathcal{D}'(M, \mathcal{L}) \mid x = z|_M \text{ for some } z \in \mathcal{D}'(X, \mathcal{L})\}.$$

An element of $\mathcal{T}(M, \mathcal{L})$ is called a tempered \mathcal{L} -distribution.

For a subset $M_0 \subset M$, put $\mathcal{D}'_{M_0}(U, \mathcal{L}) = \{x \in \mathcal{D}'(U, \mathcal{L}) \mid \text{supp } x \subset M_0\}$ and $\mathcal{T}_{M_0}(M, \mathcal{L}) = \{x \in \mathcal{T}(M, \mathcal{L}) \mid \text{supp } x \subset M_0\}$. Assume that M_0 is a closed submanifold of M . Then dualizing the restriction map $C_c^\infty(M, \mathcal{L}) \rightarrow C_c^\infty(M_0, \mathcal{L})$, we have an injective map $\mathcal{D}'(M_0, \mathcal{L}) \rightarrow \mathcal{D}'_{M_0}(M, \mathcal{L})$. This map also implies $\mathcal{T}(M_0, \mathcal{L}) \rightarrow \mathcal{T}_{M_0}(M, \mathcal{L})$. Using these maps, we regard $\mathcal{D}'(M_0, \mathcal{L})$ and $\mathcal{T}(M_0, \mathcal{L})$ as a subspace of $\mathcal{D}'_{M_0}(M, \mathcal{L})$ and $\mathcal{T}_{M_0}(M, \mathcal{L})$, respectively.

§A.2. \mathcal{L} -distributions with support in a subspace

Let M be a Euclidean space $\mathbb{R}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n\}$ and M_0 a subspace \mathbb{R}^{n-m} of M defined by the equation $x_1 = \dots = x_m = 0$. Assume that M has a compactification X . Let E_1, \dots, E_m be vector fields on M such that:

- (1) for all $\varphi \in C^\infty(M)$ we have $(E_i \varphi)|_{M_0} = (\frac{\partial}{\partial x_i} \varphi)|_{M_0}$.
- (2) A space $\sum_{i=1}^m \mathbb{C} E_i$ is a Lie algebra.

Set $D_i = \frac{\partial}{\partial x_i}$. The condition (1) implies that $D_i T = E_i T$ for all $T \in \mathcal{D}'(M_0, \mathcal{L})$. Put $U_n(E_1, \dots, E_m) = \sum_{k_1 + \dots + k_m \leq n} \mathbb{C} E_1^{k_1} \dots E_m^{k_m}$ and $U(E_1, \dots, E_m) = \sum_n U_n(E_1, \dots, E_m)$. Then the algebra $U(E_1, \dots, E_m)$ is isomorphic to the universal enveloping algebra of $\sum_{i=1}^m \mathbb{C} E_i$. For $\alpha = (\alpha_1, \dots, \alpha_m)$, put $E^\alpha = E_1^{\alpha_1} \dots E_m^{\alpha_m}$ where $E_i^0 = 1$.

Lemma A.1. *Let E'_1, \dots, E'_m be vector fields on M which satisfy the same conditions of E_1, \dots, E_m . We have*

$$E^\alpha T \in (E')^\alpha T + U_{|\alpha|-1}(E'_1, \dots, E'_m) \mathcal{T}(M_0, \mathcal{L})$$

for $T \in \mathcal{T}(M_0, \mathcal{L})$ and $\alpha \in \mathbb{Z}_{\geq 0}^m$.

PROOF. First we remark that if the order of differential operator P is less than or equal to k , then we have $P(\mathcal{T}(M_0, \mathcal{L})) \subset U_k(D_1, \dots, D_m) \mathcal{T}(M_0, \mathcal{L})$. Take $P \in U_{k-1}(E_1, \dots, E_m)$. Then we have

$$\begin{aligned} E_i P T &= [E_i, P] T + P E_i T = [E_i, P] T + P D_i T = [E_i - D_i, P] T + D_i P T \\ &\in D_i P T + U_{k-1}(D_1, \dots, D_m) \mathcal{T}(M_0, \mathcal{L}) \end{aligned}$$

since the order of $[E_i - D_i, P]$ is less than or equal to k . Hence, using induction on $|\alpha|$, we have $E^\alpha T \in D^\alpha T + U_{|\alpha|-1}(D_1, \dots, D_m) \mathcal{T}(M_0, \mathcal{L})$.

Hence we have $U_k(E_1, \dots, E_m) \mathcal{T}(M_0, \mathcal{L}) \subset U_k(D_1, \dots, D_m) \mathcal{T}(M_0, \mathcal{L})$. We prove $U_k(E_1, \dots, E_m) \mathcal{T}(M_0, \mathcal{L}) = U_k(D_1, \dots, D_m) \mathcal{T}(M_0, \mathcal{L})$ by induction on k . If $k = 0$ then the claim is obvious. Assume that $k > 0$ then the above equation and inductive hypothesis imply that if $|\alpha| = k$, then we have $D^\alpha T \in E^\alpha T + U_{k-1}(E_1, \dots, E_m) \mathcal{T}(M_0, \mathcal{L})$. Hence we have $U_k(D_1, \dots, D_m) \mathcal{T}(M_0, \mathcal{L}) \subset U_k(E_1, \dots, E_m) \mathcal{T}(M_0, \mathcal{L})$.

The same formulas hold for E'_1, \dots, E'_m . Hence, we have

$$\begin{aligned} E^\alpha T &\in D^\alpha T + U_{|\alpha|-1}(D_1, \dots, D_m) \mathcal{T}(M_0, \mathcal{L}) \\ &= (E')^\alpha T + U_{|\alpha|-1}(D_1, \dots, D_m) \mathcal{T}(M_0, \mathcal{L}) \\ &= (E')^\alpha T + U_{|\alpha|-1}(E'_1, \dots, E'_m) \mathcal{T}(M_0, \mathcal{L}). \end{aligned}$$

□

Proposition A.2. *A map $\Phi: U(E_1, \dots, E_m) \otimes \mathcal{T}(M_0, \mathcal{L}) \rightarrow \mathcal{T}_{M_0}(M, \mathcal{L})$ defined by $P \otimes T \mapsto PT$ is isomorphic.*

PROOF. First we prove that Φ is injective. Let $\sum_{\alpha \in \mathbb{Z}_{\geq 0}^m} E^\alpha \otimes T_\alpha$ (finite sum) be an element of $U(E_1, \dots, E_m) \otimes \mathcal{T}(M_0, \mathcal{L}|_{M_0})$. Set $\bar{T} = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^m} E^\alpha T_\alpha$ and assume that $T = 0$. Put $k = \max\{|\alpha| \mid T_\alpha \neq 0\}$. We prove that $k = -\infty$. Assume that $k \geq 0$. By Lemma A.1, if $|\alpha| = k$ then $E^\alpha T_\alpha \in D^\alpha T_\alpha + U_{k-1}(D_1, \dots, D_m) \mathcal{T}(M_0, \mathcal{L})$. There exist T'_α such that $\sum_{\alpha \in \mathbb{Z}_{\geq 0}^m} E^\alpha T_\alpha = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^m, |\alpha| < k} D^\alpha T'_\alpha + \sum_{\alpha \in \mathbb{Z}_{\geq 0}^m, |\alpha| = k} D^\alpha T_\alpha$. Fix $\beta \in \mathbb{Z}_{\geq 0}^m$ such that $|\beta| = k$ and $f \in C^\infty(M_0)$ with values in \mathcal{L} . Define a function φ on M by

$\varphi(x_1, \dots, x_n) = x_1^{\beta_1} \cdots x_m^{\beta_m} f(0, \dots, 0, x_{m+1}, \dots, x_n)$. Then we have $0 = \langle T, \varphi \rangle = \beta_1! \cdots \beta_m! \langle T_\beta, f \rangle$. Since f is arbitrary, we have $T_\beta = 0$ for all β such that $|\beta| = k$. This is a contradiction.

We prove that Φ is surjective. Take an open covering $X = \bigcup_{\lambda \in \Lambda} U_\lambda$ such that U_λ is isomorphic to a Euclidean space and $\overline{M_0} \cap U_\lambda$ is a subspace of U_λ where $\overline{M_0}$ is a closure of M_0 in X . Since X is compact, we may assume this is a finite covering. It is sufficient to prove that the map Φ is surjective on $U_\lambda \cap M$. However the surjectivity of Φ on $U_\lambda \cap M$ follows from [KV96, (2.8)]. \square

§A.3. Distributions on a nilpotent Lie group

Let N be a connected, simply connected nilpotent Lie group. Put $\mathfrak{n} = \text{Lie}(N)_\mathbb{C}$. Then the exponential map $\exp: \text{Lie}(N) \rightarrow N$ is a diffeomorphism. A structure of a vector space on N is defined by the exponential map. Let $\mathcal{P}(N)$ be a ring of polynomials with respect to this vector space structure (cf. Corwin and Greenleaf [CG90, §1.2]).

Let \mathcal{L} be a vector bundle on N whose fiber is V and assume that \mathcal{L} is trivial on N , i.e., $\mathcal{L} = N \times V$. Fix a Haar measure dn on N . For $F \in C^\infty(N, V')$, we define a distribution $F\delta$ by $\langle F\delta, \varphi \rangle = \int_N F(n)(\varphi(n))dn$. Then we regard $C^\infty(N, V')$ as a subspace of $\mathcal{D}'(N, \mathcal{L})$. Let $\mathcal{P}_k(N)$ be the space of polynomials whose degree is less than or equal to k . Put $\mathcal{P}(N) = \bigoplus_k \mathcal{P}_k(N)$.

Take a character η of \mathfrak{n} . Then η can be extended to the \mathbb{C} -algebra homomorphism $U(\mathfrak{n}) \rightarrow \mathbb{C}$ where $U(\mathfrak{n})$ is the universal enveloping algebra of \mathfrak{n} . We denote this \mathbb{C} -algebra homomorphism by the same letter η . Let $\text{Ker } \eta$ be the kernel of the \mathbb{C} -algebra homomorphism η . For $X \in \mathfrak{n}$ and C^∞ -function ψ , put $(X\psi)(n) = \frac{d}{dt} \psi(\exp(-tX)n)|_{t=0}$.

The algebraic tensor product $C_c^\infty(N) \otimes V$ is canonically identified with a linear subspace of $C_c^\infty(N, \mathcal{L})$ via $\varphi \otimes v \mapsto (x \mapsto \varphi(x)v)$. As in [KV96, (2.1)], this is a dense subspace of $C_c^\infty(N, \mathcal{L})$.

Proposition A.3. *For all $k \in \mathbb{Z}_{>0}$, there exists a positive integer l such that, if $T \in \mathcal{D}'(N, \mathcal{L})$ satisfies $(\text{Ker } \eta)^k T = 0$ then $T \in (\mathcal{P}_l(N) \otimes V')\delta$.*

Conversely, for all $l \in \mathbb{Z}_{>0}$ there exists a positive integer k such that $(\text{Ker } \eta)^k (\mathcal{P}_l(N) \otimes V')\delta = 0$.

PROOF. Fix a basis $\{e_1, \dots, e_n\}$ of $\text{Lie}(N)$. The map $\mathbb{R}^n \rightarrow N$ defined by $(x_1, \dots, x_n) \mapsto \exp(x_1 e_1 + \cdots + x_n e_n)$ is an isomorphism. Using this map, we introduce a coordinate (x_1, \dots, x_n) of N . If $V = \mathbb{C}$ then this proposition is well-known. Fix $v \in V$ and consider an ordinal distribution

$T_v: \varphi \mapsto \langle T, \varphi \otimes v \rangle$ for $\varphi \in C_c^\infty(N)$. If T satisfies $(\text{Ker } \eta)^k T = 0$, then T_v satisfies $(\text{Ker } \eta)^k T_v = 0$. Hence for some l , $T_v = \sum_{\alpha_1 + \dots + \alpha_n \leq l} (x_1^{\alpha_1} \cdots x_n^{\alpha_n} \otimes c_{v, \alpha_1, \dots, \alpha_n}) \delta$, where $c_{v, \alpha_1, \dots, \alpha_n} \in \mathbb{C}$. The map $v \mapsto c_{v, \alpha_1, \dots, \alpha_n}$ is continuous linear. Hence it defines an element of V' . We denote this element by $v'_{\alpha_1, \dots, \alpha_n}$. Then for $\varphi \in C_c^\infty(N)$ and $v \in V$ we have $\langle T, \varphi \otimes v \rangle = \langle (\sum_{\alpha_1 + \dots + \alpha_n \leq l} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \otimes v'_{\alpha_1, \dots, \alpha_n}) \delta, \varphi \otimes v \rangle$. Since $C_c^\infty(N) \otimes V$ is dense in $C_c^\infty(N, \mathcal{L})$, we have $T = (\sum_{\alpha_1 + \dots + \alpha_n \leq l} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \otimes v'_{\alpha_1, \dots, \alpha_n}) \delta$.

We prove the second part of the proposition. For $X \in \mathfrak{n}$, $f \in \mathcal{P}_l(N)$ and $v' \in V'$, we have $X((f \otimes v') \delta) = ((Xf) \otimes v') \delta$. Hence we may assume that $V = \mathbb{C}$. In this case, the claim is well-known. \square

Corollary A.4. *Let $T \in \mathcal{D}'(N, \mathcal{L})$. Assume that there exists a positive integer k such that $(\text{Ker } \eta)^k T \in (\mathcal{P}(N) \otimes V') \delta$. Then we have $T \in (\mathcal{P}(N) \otimes V') \delta$.*

PROOF. By the second part of Proposition A.3, there exists a positive integer k' such that $(\text{Ker } \eta)^{k'} T = 0$. Hence we have $T \in (\mathcal{P}(N) \otimes V') \delta$ by the first part of Proposition A.3. \square

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